ELEMENTARY MECHANICS.

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PREFACE.

A FEW years' experience, both as Student and as Tutor, has led me to a grateful appreciation of the method of teaching which is at present pursued by our University; one of the highest aims of which is to promote habits of rapid and accurate thought, and in this it is eminently successful. But the nature of the training adopted for this purpose has proved injurious to the style of our Mathematical Literature, more especially in the department of Natural Philosophy. Our educational books of Natural Science, in seeking to fulfil the requirements of the Senate-House, have become mere collections of definite propositions; generally containing, it is true, all the known facts and the mathematical deductions dependent upon them which appertain to the respective subjects, but seldom offering the smallest relief to the barrenness which is incidental to such compositions: in this shape they have lost all the fulness and freshness of life; all popular illustration, all freedom of expression has been denied them; they have been deprived, in fact, of nearly all the charms that usually render the pursuit of science fascinating to the young.

This evil, great as it must be confessed to be, and difficult of complete removal, yet seems easily susceptible of alleviation. If it be found that the mathematical framework of a

science cannot be filled up with a sufficient portion of illustration and remark, without danger of forming a work too bulky and too difficult of reference for the time-pressed Senate-House student, could not the two parts be with advantage separated? the first might then appear under the rigid form of a Syllabus, and the second could be sought in books, whose style should, more nearly than it does at present, approach that in which our continental neighbours are accustomed to write upon Physics.

It is often urged that it is better to leave each student to construct his own syllabus than to present him with a printed abstract, the very conciseness of which must be repulsive to him: such may be the theory, but if so, it would seem not to work well in practice; for with rare exceptions the syllabus which an Undergraduate uses, is the production of either his public or his private Tutor. But it is rather with the view of improving the style of our mathematical writing than on account of the specific utility of a published syllabus, that I conceive the system advocated should be more widely extended than it hitherto has been.

So strongly did these opinions influence me, that in the early part of this year I offered a Syllabus of Statics for publication. My aim was to present a complete scheme of the subject to the student, wherein the most important propositions, and those which have met with some little neglect at the hands of our University writers, should be illustrated both verbally and graphically; in some instances the definitions would have been given in a new form of words, and the sequence of propositions slightly altered. But it was intended that the marked characteristic of the work should be the copiousness of examples, solved for the most part geometrically, and accompanied by explanations of the circumstances of each case and of the principles involved in it.

A few words may here be allowed me in justice to myself to explain why, with my views remaining unchanged, I ultimately gave my work a character so inconsistent with them. Upon submitting the MS. to the consideration of some friends, who were obliging enough to undertake its revisal, and certainly qualified in no ordinary degree for the task, they were pleased to express approbation of its arrangement, but urged me to change in some measure its design; they wished me to turn it into a complete treatise upon Elementary Statics and Dynamics, considering that its simple geometrical character rendered it peculiarly adapted to the wants of Schools and the junior students of the University. I was perhaps too ready to adopt their suggestions, without measuring sufficiently my own power to carry them into effect: at any rate the patchwork that was necessary in order to complete the work of transformation is the cause of, and at the same time affords the only excuse I can offer for, the unevenness of construction which obtains throughout the book.

I am well aware of how greatly I stand in need of forbearance at the hands of those who may think it worth their while to pass judgment upon the following treatise; but while I trust that the preceding explanation is sufficient to remove all misconstruction regarding the nature of its pretensions, I hope that its many deficiencies will not render it altogether inefficient and devoid of utility.

J. B. P.

Clare Hall, Nov. 1850.

CONTENTS.

CHAPTER I.

SECTION I.

SECTION II.

SECTION III.

SECTION IV.

SECTION V.

SECTION VI.

CHAPTER II.

SECTION I.

SECTION II.

DYNAMICS.

SECTION I.

SECTION II.

SECTION III.

SECTION IV.

SECTION V.

SECTION VI.

STATICS.

CHAPTER I.

SECTION I.

INTRODUCTORY DEFINITIONS.

1. Any portion of matter having a definite form and magnitude is called a *body*.

A *particle* is a body of indefinitely small dimensions: it is sometimes called a material point. A body may be looked upon as a collection of material particles; and therefore, since all particles of the same substance may be conceived to be identical in every respect, the quantity of matter in bodies formed of the same kind of material will be proportional to the number of their constituent particles.

Of two bodies, one would be said to have greater *density*, or to be *denser* than the other, if it contained more particles of the same substance within the same space than the other.

This explanation gives but a rough idea of what is meant by the terms 'quantity of matter' and 'density,' with reference to bodies of the same kind, and none at all as to their use in the comparison of bodies of different kinds. They will, however, be the subject of more accurate definition under the head of Dynamics.

2. A body is said to be *rigid* when its constituent particles never change their relative positions in consequence of any change of circumstances to which they may be submitted.

When the contrary is the case, it is *not rigid;* it is capable of *compression* or *expansion.* If such a body, when released from the circumstances which have caused the compression or expansion, always return again to its original shape and condition, it is said to be *elastic.*

3. When the constituent particles of a body occupy different positions in space at successive instants of time, the body is said to be in motion; otherwise it is at rest.

4. DEF. *Any cause which produces or tends to produce a change in a body's state of rest or motion, is called Force.*

5. When several forces act simultaneously upon a body, their individual tendencies to produce motion may so counteract each other that no motion shall ensue; these forces are then said to be in equilibrium.

6. There are three things to be considered with reference to a force which acts upon a material particle:—

First, the position of that particle which is also called the point of application of the force.

Ex. If a body be pushed along a table by means of a pointed rod, the force which the rod exerts on the body so as to produce motion, is applied only to that particle of the body which it touches, and which is therefore called its point of application.

Second, the direction of the force, that is the direction in which it would make the particle begin to move, if its tendency to produce motion were not in any way counteracted by the simultaneous action of any other forces upon the particle.

Ex. Thus if a small heavy particle be suspended by a thread, as soon as the thread is severed it will begin to fall towards the ground under the action of the force of the earth's attraction alone; the direction of this incipient motion is the direction of the force alluded to.

Again, suppose a small particle of steel-filing to be gummed to a piece of cork which is floating in a bason of water and

fastened by a thread to the edge of the bason; if a magnet be held close to the surface of the fluid in front of the cork no effect will appear to be produced as long as the thread remain firm, but if the thread be cut, the particle of steel will instantly move in a straight line towards the magnet; this straight line is the direction of the force which the attractive power of the magnet exerted upon the steel.

Third, the intensity of the force, or its magnitude in reference to some other force which is taken as a measure, or, in technical language, as a unit. When any subject of which we speak has length, breadth, or thickness, it is easy to comprehend in what sense it can be said to consist of so many, say twenty, units of its particular kind; it is actually made up of twenty parts, each of which is equal in size to the assumed unit: thus, a line twenty inches long can be cut into twenty pieces, each of which is coincident in length with that which we call an inch. A carpet containing twenty square feet can be divided into twenty parts, each coinciding exactly with the area of a square foot, and so on. Similarly, when we speak of a force P, we mean that it is the same thing as a force made up of P forces acting together, each of which is equal to that force which we take as the measure or unit of force, whatever it may be; *i.e.* it is a force which when applied to a particle would produce the same statical effect as, or, in other words, would be equal to, the P equal forces applied together upon the particle and acting in the same direction.

It is only necessary therefore to define what is meant by the *statical* equality of forces.

7. Def. When two forces applied to the same particle, acting in the same straight line but in opposite directions, keep the particle at rest, they are said to be equal.

In the example given above both the magnet and the thread exert a force upon the steel point, their directions lie in the same straight line, but their tendencies to produce motion are exactly opposite: as the particle remains at rest, the forces are by definition equal.

Again, if two men pulling with their greatest strength, one at each end of the same cord, do not move each other, the forces which they respectively exert on the cord are equal. If now these two pull together at the same end of the cord and a third man pulling alone at the other end can just balance them, the force of this last is statically equal to the united force of the first two, and therefore double of either of them.

In this way all forces, whatever may be the nature of the cause which originates them, become capable of measurement and can be expressed in numbers which indicate what multiples they are of the unit of force. It is perfectly arbitrary with us what particular force we take as the unit wherewith to measure all others; we consult convenience only in choosing it. It is sufficiently clear, that although by altering the unit we should alter the numbers expressing any given forces, still the ratios between those numbers would remain unchanged.

8. The three things to be considered with reference to a force, and which have just been mentioned as sufficient and necessary to distinguish forces amongst themselves, are exactly those which serve to define any particular straight line, *i.e.* point of beginning, direction, and magnitude; hence straight lines may be employed very advantageously for the purpose of representing forces, and Geometry may be made auxiliary to the solving of Statical problems.

Whenever, therefore, in Statics a given straight line is said to represent a force, it must be understood that the first point of the line is the point of application of the force, the direction of the line is the direction of the force, and that the force contains as many units of force as the lines does of length.

9. It may be as well here to mention some of the various causes which produce force. Those with which we are principally concerned are:—

a. *Gravity*, or the attraction of the earth for all particles of matter. The force exerted by it is generally of different magnitude for different bodies, varying in some degree with

their size and texture; but it is found to be constantly of the same magnitude for the same body at the same place on the earth's surface; it is termed its weight. Its direction is always perpendicular to the surface of still water, and is usually designated as the *vertical line*. A body when free and left to the action of gravity alone, will fall to the surface of the earth: a force sufficient to sustain it, *i.e.* to keep it at rest in opposition to the force of gravity, will, by the definition already given, be equal to its weight.

β. *Attractions* of any other kind existing between particles of matter.

Ex. Electrical and magnetic attractions and repulsions.

γ. *Tensions* of strings and of fine rods.

A body may be moved along a direction or held in a certain position by means of a rod or string, one of whose extremities is attached to it, and the other communicates with some cause of force, such as the hand of a person: in such a case force is said to be transmitted by the rod or string from the hand to the body; it is called the tension of the rod or string, and its point of application to the body is the extremity of the rod or string.

The tension at any point of a rod or string is, more properly speaking, the force exerted by the particle at that point upon the particle next succeeding, by means of which the force is transmitted from one extremity to the other, as before mentioned: it may therefore be conceived to vary from point to point; it will always do so when other forces act upon the string than those applied at its extremities. Thus, if a barge be towed along by the aid of a *heavy* rope, the force applied to the barge by one end of the rope is not necessarily the same as that applied by the horse at the other, and both will differ from the tensions at different points of the rope. The reason of this is, that each particle of the rope is acted upon by the force of gravity as well as by the tension of the preceding and succeeding particles.

A string is defined to be *flexible* when the direction of the tension at every point of it is always the same as that of the string at that point.

If the string be also without weight, the tension at *every* point of it will be the *same.*

An imperfectly flexible string only differs from a perfectly flexible one, in that it requires time in order to assume its position of equilibrium, when forces are applied to it. When this position is once attained, the tension at every point of the string is, as before, in the direction of the string. As therefore in all Statical questions the position of equilibrium is supposed to be attained, strings are for us always flexible, and whether they lie in a straight line or in a curve, the tension at all points of them is *along* the string.

Hence, on the contrary, rods may exert a force transversal to their length, for if they were incapable of doing so they would fall under our definition of flexible strings. Generally the force exerted or transmitted by a rod must be a subject for determination, both as regards its magnitude and direction in any given case: but if the rod be without weight and its extremities capable of turning about *free hinges*, the direction of its tension must be *along its length.* This tension may be either a pulling or a pushing force, while the tension of a string must be always a pulling force.

8. *Pressure* of one surface upon another in contact with it.

This force is generally a subject for determination in every problem. It acts equally, but in opposite directions, upon both bodies at the point of contact, tending to push each away from the other body. The surfaces of the bodies are defined to be perfectly *smooth* when the line of action of this force always coincides with their common normal: if it does not necessarily do so, they are termed rough.

10. The unit which it is found most convenient to employ for measuring forces is what is commonly called a pound weight; *i.e.* it is a force equal, according to the above defini-

tion of equality, to just so much of the force of gravity as is called into action by matter whose weight is one pound: thus, when we speak of a force of P pounds, we mean a force equal to P such units; or, in other words, a force which will keep in equilibrium P forces acting together, each equal to the force of gravity exerted upon matter whose weight is known as one pound.

11. In Statics bodies are only looked upon as furnishing points whereon forces may act: their weight is the apparent result of an external force, that of gravity, acting upon each of their constituent points. It will be seen hereafter that its effect in producing or disturbing equilibrium can be easily found in any given case. The rigidity or non-rigidity of the bodies is capable of being expressed by means of geometrical relations between the points upon which the forces act, or by other forces applied to these points of such a nature as to insure the geometrical relations required.

It is the great object of Statics to discover what must be the relations between a given set of forces as regards their magnitude and direction, and the system of points upon which they act, in order that these points may be kept at rest; or, in other words, in order that equilibrium may be produced in the system.

To this end it is necessary to investigate under what circumstances and in what manner it is possible to replace a given set of forces by another of a simpler or more convenient kind, without affecting the state of equilibrium of the system.

SECTION II.

FORCES ACTING AT THE SAME POINT IN ONE PLANE.

12. When a material particle is subjected to the simultaneous action of several forces, say of P_1, $P_2 ... P_n$, such as do not keep each other in equilibrium, it will begin to move in some definite direction in a definite manner; it is clear that this particular motion might in every case be produced by some one force alone: in Statics the movement does not take place, it is counteracted by some other force or forces; but the tendency to it exists and is the sole effect of these combined forces P_1, $P_2 ... P_n$. It is also clear that the same tendency would be produced by the before-mentioned single force: such a single force is called the *resultant* of the forces P_1, $P_2 ... P_n$, and these last are called the *components* of the *resultant*, or more often the *resolved parts* of the resultant in their particular directions.

From this definition it is evident that the single force which acting upon the particle would keep the forces P_1, $P_2 ... P_n$ in equilibrium, must be equal to their resultant in magnitude, but opposite to it in direction. This consideration often affords an easy practical method of finding the resultant of a given set of forces.

It must be remembered that the resultant of any given forces never has any real existence, but is simply that force which might be substituted for the given set, without altering the effect produced by them towards preserving the equilibrium of the system.

In the more extended case when the forces do not all act at the same point of the body or in the same plane, the term

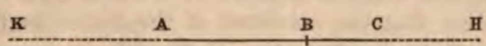
resultant is used in the same sense as that explained above, but it does not follow as a matter of course that a single resultant shall exist; two forces are sometimes necessary in order to produce the effect of the set in maintaining equilibrium.

13. The first great purpose of Statics is to obtain rules for finding the resultant force or forces in any proposed case.

If the components all act in the same straight line and in the same direction, it is clear from the definition of the measure of forces, that they form a single force equal to the sum of them. And if some of them act in the other direction, the whole set is equivalent to the sum of the former diminished by the sum of the latter; or if those which act in the first direction be considered positive, and those in the last negative, the single resultant force is equal to the algebraical sum of the components.

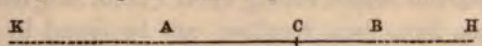
Suppose the forces P and Q to act at a point A in the same straight line KH, and

First, let them both act in the same direction towards H,



the total or resultant force upon A is $P+Q$ towards H: take AB equal to P units of length, and BC equal to Q units, then AC contains $P+Q$ units; and AB, BC, AC, of which the last is equal to the sum of the two first, correctly represent the forces P and Q, and their resultant P and Q.

Next suppose the second force Q to act towards K, this will evidently keep in equilibrium a portion of the force P equal



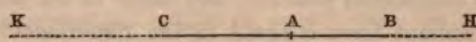
to Q, and these two will neutralize each other, thus leaving a force $P-Q$ as the resultant force acting in the direction of H: take AB as before, equal to P units of length, and cut off from it a length BC equal to Q units; then AC will contain $P-Q$ units, and will correctly represent the resultant of the forces P and Q acting upon A in the directions H and K respectively.

If we make the convention that lines representing forces shall be positive when measured in one direction and negative when measured in the opposite, both the above results can be put under one form: thus, let p and q represent the actual numerical lengths of AB and BC; then in the first case $BC = q$, in the second $BC = -q$, and therefore

$$\left.\begin{aligned} AC &= p + q \text{ in first case} \\ &= p - q \text{ in second case} \end{aligned}\right\} = AB + BC \text{ in both,}$$

that is, the line representing the resultant of the forces is equal to the sum of the lines representing the forces themselves.

If in the second case q were greater than p, we should have $AC = -(q - p)$ a negative quantity; it *ought* therefore by our

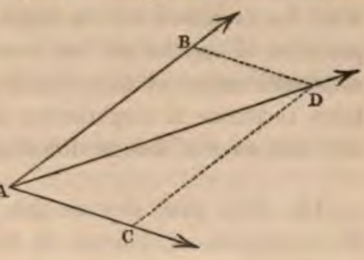


convention to be measured in the opposite direction to AB: now by reference to the figure, we observe that since BC is numerically greater than AB, C must lie between A and K, and therefore AC *is* measured in a direction opposite to AB. It is also clear that the resultant of the forces in this case is a force $Q - P$ in the direction of K. Hence our convention always leads to consistent results. It is the rule which is generally adopted in expressing forces by geometrical quantities. Of course it is perfectly arbitrary which direction of any two be taken as positive, and can only be made a matter of convenience in the individual cases.

14. When the forces acting upon a point are not all in the same straight line, their resultant can be found by the aid of the proposition of the *Parallelogram of Forces*, which asserts that—

If two forces acting upon a particle be represented both in magnitude and direction by two straight lines drawn from that point, then their resultant will be represented in magnitude and direction by the diagonal, passing through that point, of the parallelogram described upon those lines.

Ex. Let AB, AC represent both in magnitude and direction two forces acting upon the point A; complete the parallelogram $ABDC$, and draw the diagonal AD; then AD represents both in magnitude and direction the resultant of the forces represented by AB and AC.



15. The proof of this proposition, which is given below, is based upon the three following assumptions:

First, the Physical Principle of the Transmission of Force, which declares that the effect of a force upon a rigid body remains unaltered at whatever point in its direction we conceive it to be applied, always supposing this point to be rigidly connected with the body.

Second, the converse of this; *i.e.* that no force can be applied with indifference as to its effect in producing or destroying equilibrium at two points of a body, unless these points be in the line of its direction.

Third, that the direction of the resultant of two equal forces applied to the same point bisects the angle between the directions of these forces.

The first of these may be termed a Law of Mechanics deduced from actual experiments; its truth may be readily seen in some simple cases: thus, if it be required to sustain a body by the aid of a string, it is indifferent at what point of the body the string be fixed, as long as this point be in the straight line in which the string lies: again, the same force is required either to push or to pull a beam in the direction of its length along the surface of water in which it floats.

It is clear that the knowledge of some such law as this is necessary to enable us to pass from the consideration of forces acting upon a point to that of forces acting upon a rigid body; for in the latter case, the particular particle of the body to which the force may be applied cannot obey the influence of the

force in exactly the same manner as if it were free; its condition must be modified by its rigid connexion with the remaining particles of the body: but our ignorance of the nature of the molecular action of the particles upon one another obliges us to have recourse to experiment in order to discover the extent and character of this modification.

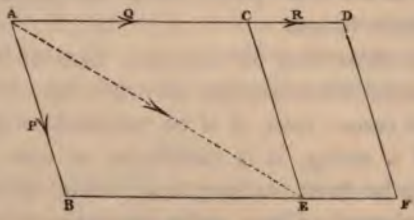
16. The *first step* in the proof of the proposition of the Parallelogram of Forces, is to shew that the diagonal of the parallelogram is the *direction* of the resultant.

It is manifest that if the forces are equal, their resultant must bisect the angle between their directions, and is therefore in this case in the direction of the diagonal of the parallelogram formed upon the lines representing the two equal forces.

Let us now assume that this first part of our proposition is true for a certain pair of forces P and Q, and also for the pair of forces P and R; we will shew that it must then be true for the pair P, and $Q+R$.

Suppose A to be the point at which this pair is applied, P acting in the direction AB and $Q+R$ in that of AD.

Take AB, AC, CD proportional to P, Q, and R respectively; then if we consider C to be rigidly connected with

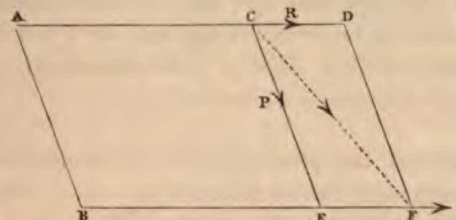


A, since it is a point in the direction of $Q+R$, we may remove the point of application of R from A to it; that is, the forces P along AB and $Q+R$ along AD, all applied at A, are statically equivalent to the forces P along AB, Q along AD both acting at A, and R along CD acting at C.

Complete the parallelograms $ABEC$, $CEFD$. By our supposition, the resultant of the pair of forces P and Q acting

at A, will lie in the direction AE, because it is the diagonal of the parallelogram described upon AB and AC which respectively represent these forces. Let this resultant be substituted for the forces, and then let its point of application be removed from A to E.

Again, at E, let us replace this resultant by its resolved parts, P and Q acting parallel to their original directions; that is, P in the direction of CE produced, and Q in the direction of EF; moreover, let the point of application of this P be removed to C, and that of Q to F, both being points in their respective directions. It is evident that, in accordance with our assumptions, our transformed system of forces, *i.e.* P and R



acting at C along CD and CE respectively, and Q at F in direction of EF produced, is equivalent in statical effect to the original system.

But the result of P and R at C lies in direction of CF, the diagonal of the parallelogram described upon the lines CE, CD which represent P and R respectively; let this resultant be substituted for them, and let also its point of application be removed from C to F.

At F let this resultant of P and R be replaced by its resolved parts P and R parallel to CD and CE respectively.

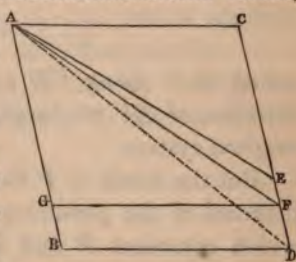
We have thus the two original forces P, acting along AB and $Q+R$ along AD, both applied at A, replaced by an equivalent pair, P acting along DF produced, and $Q+R$ along EF produced both applied at F. The resultant of these last is evidently the same in magnitude and direction as that of the first pair; and hence we come to the conclusion that it is indifferent as to statical effect, whether the resultant of the

given forces P and $Q+R$ acts at A or at F; A and F are therefore two points in its direction; and we have therefore proved that if the proposition of the parallelogram of forces be true for two forces P and Q, and also for two forces P and R, it must necessarily be true for P and $Q+R$.

But the proposition has been seen to be true for any pair of equal forces as p and p: and also for a second pair p and p, and therefore by the preceding for p and $2p$: again, since it is true for p and p, and also p and $2p$, it is therefore true for p and $3p$; similarly it may be shewn by successive deduction to be true for p and mp where m is any whole number.

Since it is true for mp and p, and also for mp and p, it is therefore true for mp and $2p$; and by proceeding as before, it may be shewn to be true for mp and np when m and n are any whole numbers; *i.e.* since p is indeterminate, it is true for any commensurable forces whatever.

17. To extend the proposition to incommensurable forces, let AB, AC represent two such forces acting simultaneously at A. Complete the parallelogram $ABDC$.

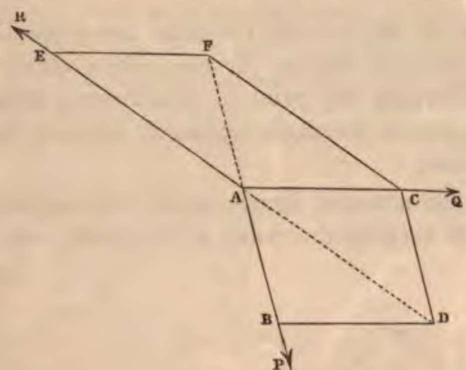


If AD be not the direction of the resultant of the forces represented by AC, AB, let some other line as AE be it, cutting CD in E. Suppose AC to be divided into a number of equal parts each less than ED, and suppose distances of the same magnitude to be marked off along CD; then one of these divisions must necessarily fall between E and D, let it be F; complete the parallelogram $ACFG$. Then if we conceived two commensurable forces to act at A, which should be represented by the lines AC, AG, the direction of their resultant would be AF, and it would lie farther away from AC than AE, the direction of the resultant of the forces represented by AC and AB, does, although AG is less than AB; which is evidently absurd. Hence AE is not the direction of the resultant of the given forces. In like manner it can be shewn that no other

line except AD is in that direction, and therefore the proposition is true for *incommensurable* as well as for *commensurable* forces.

The *second step* in our proof is to shew that the proposition is true as regards the magnitude of the resultant.

As before, let AB, AC represent the two forces P and Q acting at the point A; and let AE represent a third force R, which also acting at A will keep P and Q in equilibrium; then, by Art. (12), R must be equal and opposite to the resultant of P and Q. For the same reason P must be equal and opposite to the resultant of Q and R; therefore, completing the parallelograms $BACD$, $CAEF$, we must have DA in the same straight line with AE, and BA with AF: hence



$AFCD$ is a parallelogram, and AD equals FC, also FC is by construction equal to AE; therefore $AD = AE$, in other words AD, the diagonal of the parallelogram described upon AC, AB which represent the forces P and Q, represents a force which is equal in magnitude to the resultant of P and Q. Q. E. D.

18. From the proposition of the *Parallelogram of Forces* follows immediately the proposition of the *Triangle of Forces*:

That if three forces acting together upon a point be in equilibrium, they are respectively proportional to the three sides taken in order of the triangle formed by drawing lines parallel to their directions. And conversely, that if three forces acting at a point

be proportional to and in the direction of the respective sides of a triangle taken in order, they must be in equilibrium.

This is in fact only another method of wording the parellelogram of forces. For that asserts that three forces acting upon a point, keeping each other in equilibrium, are proportional to the two sides and diagonal of a parallelogram whose directions are parallel to those of the forces, and conversely. But the sides and diagonal form a triangle, as may be seen by reference to triangle ACD of the last figure; and it is therefore the same thing whether we say of three straight lines that they are the sides and diagonal of a parallelogram, or that they will form a triangle. The *parallelogram* and the *triangle* of forces are therefore identical propositions.

19. Since in any triangle the sides are proportional to the sines of the opposite angles, our proposition shews us, that of three forces keeping any point in equilibrium, each is proportional to the sine of the angle contained between the directions of the other two.

Thus, in the annexed figure, which represents three forces P, Q, and R keeping the point A in equilibrium, if ABC be

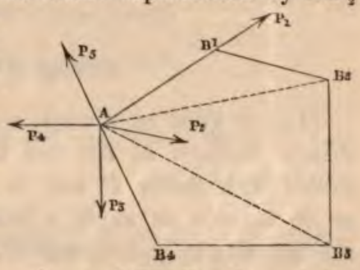
a triangle having its sides parallel to the forces, we have by

triangle of forces,

$$P : Q : R :: AB : BC : AC,$$
$$:: \sin ACB : \sin BAC : \sin ABC,$$
$$:: \sin QAC : \sin BAC : \sin PBC,$$
$$:: \sin QAR : \sin RAP : \sin PAQ.$$

20. *In a similar manner, if any number of forces acting upon a point are parallel and proportional to the sides taken in order of a polygon, they must keep each other in equilibrium.*

Let $AB_1B_2B_3B_4$ be the polygon to whose sides the forces $P_1P_2..P_5$ acting at A, are respectively parallel and proportional. Join AB_2, then the triangle AB_1B_2 has its sides AB_1, B_1B_2 respectively parallel and proportional to the forces P_1P_2; therefore the third side AB_2 is in the direction of, and proportional to, their resultant. Similarly AB_3 is in the direction of, and proportional to, the resultant of the forces represented by AB_2 and B_2B_3; but B_2B_3 is parallel and proportional to P_3, and AB_2 is parallel and proportional to resultant of P_1 and P_2, therefore AB_3 is parallel and proportional to the resultant of P_1, P_2, and P_3. By proceeding in the same way it could be shewn that AB_4 is parallel and proportional to the resultant of P_1, P_2, P_3, and P_4, but the same line is by hypothesis parallel and proportional to the last force P_5 in a direction opposite to AB_4. Therefore the resultant of the four first forces is equal and opposite to the fifth and last force, hence the five forces $P_1P_2...P_5$ keep each other in equilibrium.



In the same way, this proposition, which is called the *polygon of forces*, could be shewn to be true of any number of forces acting at a point. It is evidently not necessary that the forces should lie in one plane.

The converse of the polygon of forces cannot, as the triangle of forces, be affirmed; for parallel-sided polygons have not necessarily their sides in the same proportion, whereas triangles have.

21. The Parallelogram of Forces enables us to substitute for a given force acting upon a particle its two resolved parts in two given directions.

The same may be done with each of any number of given forces acting in the same plane upon a point. Hence any such system of forces may be resolved into known components acting in two given directions.

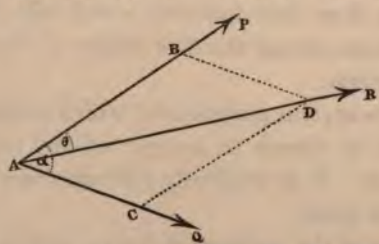
22. A further application of the parallelogram of forces leads us to the conclusion, that if the given forces be in equilibrium, then must their components in any two given directions separately vanish.

From this proposition is deduced the analytical method of forming the "Equations of Equilibrium" of a material particle under the action of given forces.

Examples to Section II.

(1). A knot in a cord is rendered immoveable by a tack driven through it, the two extremities of the cord are then pulled with forces P and Q respectively, in such a manner as to include an angle α between their directions; required the magnitude and direction of the resultant force upon the tack.

Let A be the tack, AP, AQ the directions in which the two strings are pulled by the forces P and Q respectively;



then $PAQ = \alpha$. In AP, AQ take AB, AC respectively proportional to P and Q; complete the parallelogram, and draw

the diameter AD. Then since the strings exert forces P and Q respectively upon the tack (Art. 9, γ), and AB, AC have been taken proportional and parallel to them, therefore AD is by the parallelogram of forces proportional and parallel to their resultant; let the resultant equal R, and let BAD equal θ. Hence

$$\frac{R}{P} = \frac{AD}{AB} = \frac{\sin\alpha}{\sin(\alpha-\theta)} \text{ by triangle } ABD \ldots\ldots (1),$$

$$\frac{R}{Q} = \frac{AD}{BD} = \frac{\sin\alpha}{\sin\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

These two equations will serve to determine R and θ, which are required by the question.

Thus from (1) and (2)

$$P\sin\theta = Q\sin(\alpha-\theta),$$

$$= Q(\sin\alpha\cos\theta - \cos\alpha\sin\theta);$$

therefore $\quad \sin\theta\,(P + Q\cos\alpha) = Q\sin\alpha\cos\theta,$

or $\quad \dfrac{\sin\theta}{Q\sin\alpha} = \dfrac{\cos\theta}{P + Q\cos\alpha} = \dfrac{1}{\sqrt{\{Q^2\sin^2\alpha + (P+Q\cos\alpha)^2\}}},$

by a known form,

$$= \frac{1}{\sqrt{(P^2 + Q^2 + 2PQ\cos\alpha)}} \ldots\ldots\ldots\ldots (3).$$

Substituting from (3) in (2), we get

$$R = \sqrt{(P^2 + Q^2 + 2PQ\cos\alpha)} \ldots\ldots\ldots\ldots (4),$$

(3) and (4) give R and θ explicitly.

R might have been determined directly; for from the triangle ABD,

$$AD^2 = AB^2 + BD^2 - 2AB.BD\cos ABD,$$

or $\quad \dfrac{AD^2}{AB^2} = 1 + \dfrac{BD^2}{AP^2} - 2\dfrac{BD}{AB}\cos ABD,$

$$\frac{R^2}{P^2} = 1 + \frac{Q^2}{P^2} + 2\frac{Q}{P}\cos\alpha,$$

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

(2). A string passes round a smooth tack, and to its extremities are applied forces each equal to P; required the direction and magnitude of the resultant force upon the tack.

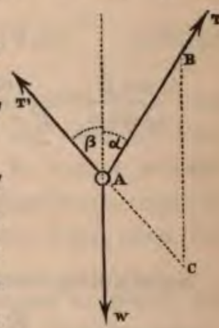
If the forces at the two extremities were not equal, it is sufficiently evident that the string would slip round the tack in the direction of the larger force.

Also each string pulls upon the tack in its own direction with a force equal to P.

Hence this case is merely a modification of the last, in which the forces P and Q are equal.

(3). A weight W is attached by a string to a ring which is supported by two strings making angles α and β respectively with the vertical; required the tensions of the strings.

Let A be the ring, T, T' the respective tensions of the strings, AT, AT' their directions.



Then the point A is kept in equilibrium by the force W acting vertically downwards, and the forces T and T' acting along AT and AT' respectively.

In AT take any point B, and draw BC parallel to AW meeting $T'A$ produced in C; then the sides of the triangle ABC are parallel to and therefore proportional to the three forces which keep A in equilibrium; hence we have

$$\frac{T}{W} = \frac{AB}{BC} = \frac{\sin\beta}{\sin(\alpha+\beta)} \qquad \text{............(1)},$$

$$\frac{T'}{W} = \frac{AC}{BC} = \frac{\sin\alpha}{\sin(\alpha+\beta)} \qquad \text{............(2)};$$

these equations determine T and T'.

If the problem had been so enunciated that the tensions of the strings were known, the angles α and β would have been given by equations (1) and (2).

(4). At what angle must two equal forces act that their resultant may be equal in magnitude to either of them separately?

Determine the angle also when the resultant is $\frac{1}{n}^{\text{th}}$ part of either.

In this question, R must first be found in terms of the forces and the angle included between them, as in example (2); if the resulting value be put equal to either force, an equation will be obtained for finding the angle.

(5). Three forces $2F$, $3F$, $4F$ act upon a point and keep it at rest; find their directions.

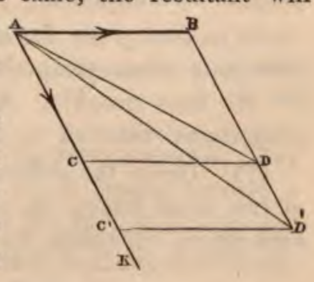
Assume the inclinations of the two last forces to the direction of the first to be α and β, as in example (3).

(6). A particle placed upon a smooth table is acted on by two forces $3P$ and $4P$ in directions perpendicular to one another. Find the magnitude and direction of a third force just sufficient to prevent the particle from moving.

It is a force exactly equal and opposite to that represented by the diagonal of the parallelogram described upon the lines representing $3P$ and $4P$.

(7). Two forces act upon a point in directions perpendicular to each other, they are in the ratio of 3 to 4, and their resultant is equal 60 lbs. Determine the forces.

(8). P and Q are two forces which act at a point and make a constant angle α with each other; shew that if Q increases from 0 whilst P and α remain the same, the resultant will constantly increase if α be acute, but will first diminish to a certain value and then increase if α be obtuse.

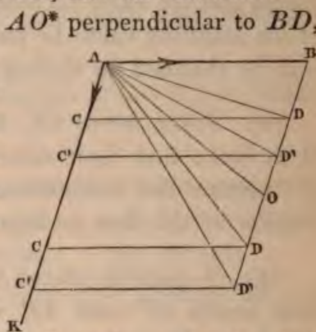


Let A be the point, and let AB represent P, and let AK be the direction of Q. Take AC to represent any value of Q, and complete the parallelogram $ABDC$, drawing

the diagonal AD; then AD represents the resultant of P and this value of Q. When Q increases to any other value represented by AC', let AD' be the resultant.

First, let BAK be acute, then ABD is obtuse, and much more is the interior angle ADD' obtuse; hence in the triangle ADD', AD' is greater than AD because it is opposite the greater angle; therefore as AC increases, AD also increases.

Next let BAK be obtuse; draw AO* perpendicular to BD, then as long as BD and BD' are less than BO, it is evident that ADD' is acute, while $AD'D$ is obtuse; hence AD is greater than AD', and therefore as AC increases up to the value BO, AD diminishes to the value AO.



But when Q has increased so that BD and BD' are greater than BO, then ADD' becomes obtuse and $AD'D$ acute, as in the first case, and hence AD' is greater than AD; or as AC increases beyond the value BO, AD also continually increases beyond the value AO.

(9). The resultant of two forces P, Q acting at an angle θ, is equal to $(2m+1)\sqrt{(P^2+Q^2)}$; when they act at an angle $\frac{\pi}{2}-\theta$ it is equal to $(2m-1)\sqrt{(P^2+Q^2)}$: shew that $\theta = \tan^{-1}\dfrac{m-1}{m+1}$.

(10). Shew that three forces, measured respectively by 5, 6, and 12 pounds, cannot be made to act upon a particle so as to produce equilibrium.

(11). Two given forces act upon a point, determine their directions when their resultant is, 1st the greatest possible, 2nd the least possible. Also find when its value is a mean proportional between its greatest and least values.

(12). Given the difference between the resultant and either of two equal forces, its components, acting at right angles to each other; determine the forces by a geometrical construction.

* The figure is badly drawn, AO should be at right angles to BD.

This is simply to find the side of a square, having given the difference between the side and the diagonal.

(13). Two forces F and F' act along the diagonals of a parallelogram and at the point of intersection of the diagonals, another force W acts in a direction perpendicular to one side: if α and α' be the angles between the diagonals and that side, shew that there will be equilibrium if

$$F \sec \alpha = F' \sec \alpha' = W \operatorname{cosec} (\alpha + \alpha').$$

(14). Three forces P, Q, R act in equilibrium at the angles of a triangle ABC, and their directions meet in the circumference of the circumscribed circle; shew that

$$P : Q : R :: BC : CA : AB.$$

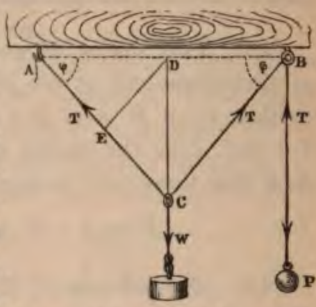
The forces must be considered to act at the point where their directions meet; the physical principle of the transmission of force (Art. 15) allows of this: the above proportion can then be easily deduced from the triangle of forces described at that point.

(15). A point in the vertex of a right-angled triangle is solicited by a number of forces represented in magnitude and direction by lines drawn to equidistant points in the base. Required the magnitude and direction of the resultant.

If the diagonal be drawn to the parallelogram described upon the two sides of the triangle which include the right angle, it will be found to coincide with the diagonal of the parallelogram described upon each pair of lines taken in order, one from each side, beginning with the extreme lines; if therefore there be n pairs of lines altogether, the resultant force of the system will be n times the force represented by this diagonal.

(16). A and B are two points in the same horizontal line; a string has one extremity fastened at A, passes through a smooth ring C which carries a weight W, runs over a smooth peg at B, and finally supports a weight P at its further ex-

tremity; required the conditions of equilibrium. Since the ring C and the peg B are both smooth, the tension of the string must be the same throughout, call it T; the point C is kept in equilibrium by the weight W acting vertically downwards and the two equal tensions T; hence it is manifest that the direction CW must bisect the angle ACB, or that the angles CAB, CBA are each equal: let ϕ be this angle. In CA take any point E, draw ED parallel to CB meeting CW produced in D, then the triangle CED has its sides parallel to, and therefore proportional to, the forces that keep C in equilibrium; therefore



$$\frac{T}{W} = \frac{EC}{DC} = \frac{\cos\phi}{\sin 2\phi} \;\ldots\ldots\ldots\ldots\; (1);$$

also since P is supported by the tension of the string

$$T = P \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

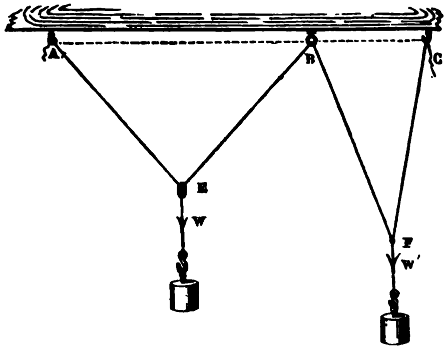
(1) and (2) serve to determine T and ϕ. It appears that equilibrium is impossible if W be greater than $2P$.

(17). Suppose the ring C to be fixed to the string so that AC is of known length; the other circumstances remaining the same as before, required the position of equilibrium.

In this case it is clear that the tensions of the strings will not necessarily be the same, they must be assumed to be T and T'; also BAC and ABC will not necessarily be equal, ABC may be called ϕ'; then in addition to equations similar to (1) and (2) of preceding question, we shall have a trigonometrical relation between ϕ and ϕ' from the triangle ABC; from these three we shall easily obtain ϕ.

(18). In the annexed figure $AEBFC$ represents a string fastened at A and C, and passing over a pulley at B. W is

a weight hanging by a smooth ring at E, and W' is attached

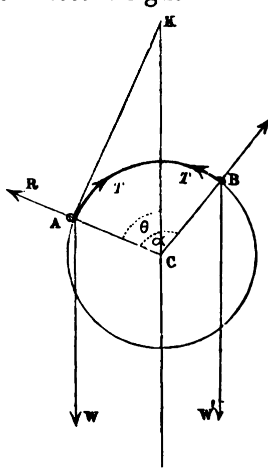


to a point F of the string.

This must be treated as two problems, the one for the equilibrium of W, which is the same as (16), and the other for the equilibrium of W', which is the same as (17). It must be remembered that since B is smooth, the tension of BF is the same as that of BE and EA.

(19). Two heavy particles connected by a string of given length rest upon a smooth vertical circle. Find their position of equilibrium.

Let A and B be the two particles whose weights are W and W' respectively, and let α be the known angle subtended by the string AB at the centre C of the circle, θ the angle made by the radius AC with the vertical.



Since the circle is smooth, the tension of the string will be the same throughout, call it T; and let R and R' be the normal reactions of the circle upon the particles A and B respectively (see Art. 9, γ).

The particle A is kept in equilibrium by its own weight W acting vertically downwards, the normal resistance R of the circle and the tension T acting tangentially.

From A draw AK at right angles to AC, meeting the vertical through C in K; then the sides of the triangle CAK are parallel, to and therefore proportional to, the forces which keep A in equilibrium; therefore

$$\frac{T}{W} = \frac{AK}{KC} = \sin\theta \dots\dots\dots\dots\dots (1),$$

$$\frac{R}{W} = \frac{AC}{KC} = \cos\theta \dots\dots\dots\dots\dots (2):$$

similarly we get from the equilibrium of B

$$\frac{T}{W'} = \sin(\alpha - \theta) \dots\dots\dots\dots\dots (3),$$

$$\frac{R'}{W'} = \cos(\alpha - \theta) \dots\dots\dots\dots\dots (4).$$

These four equations serve to determine the four unknown quantities R, R', T, θ.

From (1) and (3)

$$W' \sin(\alpha - \theta) = W \sin\theta,$$

or $$W'(\sin\alpha\cos\theta - \cos\alpha\sin\theta) = W\sin\theta;$$

$$\therefore \frac{\cos\theta}{W + W'\cos\alpha} = \frac{\sin\theta}{W'\sin\alpha} = \frac{1}{\sqrt{\{W'^2\sin^2\alpha + (W + W'\cos\alpha)^2\}}}$$

$$= \frac{1}{\sqrt{(W^2 + W'^2 + 2WW'\cos\alpha)}}.$$

(20). From this deduce the case in which one of the weights, as W, hangs completely over the cylinder.

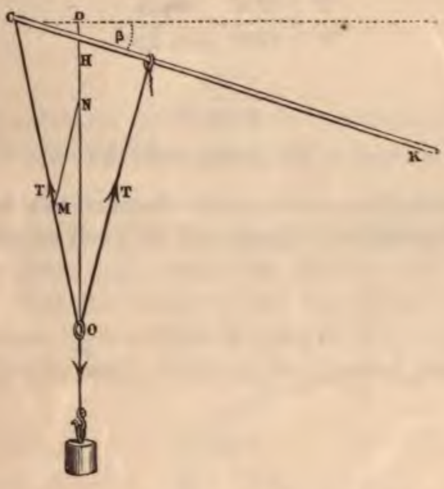
(21). A parabola stands with its axis vertical, and vertex downwards: on its outside surface rest two heavy particles connected by a string which passes over a smooth peg in the focus. Shew that if the particles are in equilibrium in any position, they are so in all.

(22). A string without weight has one extremity fixed to a point C, the other is attached to a ring capable of sliding along a smooth fixed rod, which also passes through C. Shew

that, if the rod be inclined to the horizon at an angle β, and a ring be hung upon the string carrying a weight, both parts into which the string is divided by this ring will be inclined to the vertical at an angle β.

Find the pressure of the ring upon the rod, and explain its physical meaning when β exceeds 45°.

The annexed figure represents the system. *COB* is the string, at one extremity of which is the ring *B*, capable of



sliding along the rod *CK*. *O* is the ring which slides along the string and carries the weight *W*.

O is kept in equilibrium by the weight *W* acting vertically downwards and the tension of the string in the directions *OB* and *OC*: these tensions must be the same, because the same string *COB* passes through the smooth ring *O*; call them *T*; then the vertical line or direction of *W* bisects the angle *COB*; let it meet the rod in *H* and the horizontal line through *C* in *D*.

Because the rod *CK* is smooth, the force it exerts upon the ring *B* must be at right angles to its length (Art. 9, δ); but this force counteracts the tension of *OB*, therefore the direction of *OB* is perpendicular to *CK*. It follows that *CDH* and *HOB*

are similar triangles, and therefore the angle

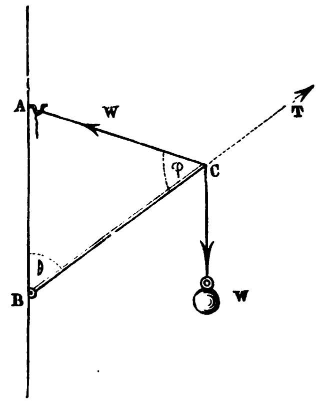
$$BOH = \text{angle } DCH = \beta,$$
$$= \text{also } COD.$$

In OC take OM to represent the tension of the string; draw MN parallel to OB meeting OD in N; then the triangle MON has its sides respectively parallel to, and therefore proportional to, the forces that keep each other at equilibrium at O.

Hence $$\frac{T}{W} = \frac{OM}{ON} = \frac{\sin \beta}{\sin 2\beta},$$
$$= \frac{1}{2 \cos \beta}.$$

This force T is equal to the pressure on the rod.

(23). A string has one extremity fastened to a fixed point A, passes over the end of a smooth rod BC, and supports a weight



W; this rod has no weight, and is capable of moving freely about the extremity B, which is also in the same vertical line with A. Find the position of equilibrium.

Let $BC = a$, $BA = b$, $ABC = \theta$, $ACB = \phi$.

Since the end of the rod is smooth, the tension of the string will be the same throughout, and manifestly equal to W. Hence the point C is kept in equilibrium by the two equal forces W, and the tension, T suppose, of the rod in the direction

of its length, (Art. 9, γ) therefore this direction must bisect the angle ACW; or

$$ACB = ABC,$$

$$\phi = \theta \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also

$$\frac{a}{b} = \frac{\sin(\phi + \theta)}{\sin \phi} \ldots\ldots\ldots\ldots\ldots (2);$$

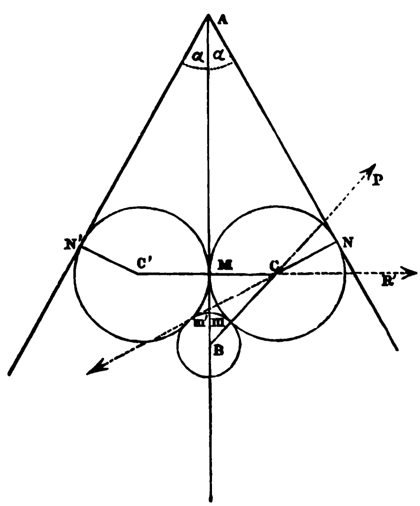
therefore

$$\left.\begin{aligned} \cos \phi &= \frac{a}{2b} \\ \phi &= 0 \\ \phi &= \pi \end{aligned}\right\}.$$

These three solutions give the three positions of equilibrium. The first is impossible when a is $> 2b$.

(24). Two equal circular discs with smooth edges placed on their flat sides in the corner between two smooth vertical planes inclined at a given angle touch each other in the line bisecting the angle. Find the radius of the least disc, which may be pressed between them without causing them to separate.

Let $2a$ be the angle between the vertical planes, CC' the



centres of the two given discs, NN' the points where they

touch the vertical planes, M where they touch each other. Also let B be the centre of the required disc touching the former in m and m'. Since the discs C and C' are similarly situated, we need only consider the equilibrium of one of them as C.

The forces acting upon this are the reaction R of the plane at N,
.... R' of disc C'.... M,
.... P...... B.... m.

These all act at C, and, as they are all indeterminate, will always be so exerted as to preserve equilibrium as long as one of them passes through the angle between the other two, *i.e.* as long as BC lies within CR and CR'. But as the radius of B is diminished, CB approaches CR, and if the radius be made small enough, will come outside CR, when the disc C would be immediately pushed out, for the three forces P, R and R' would then have a resultant tending from the angle A.

Hence the least radius that B can have consistent with equilibrium is when CB coincides with CR, or when

$$\frac{CM}{CB} = \cos a.$$

Let r be radius of given discs, x that of B, then this condition becomes

$$\frac{r}{r+x} = \cos a,$$

therefore $$x = r\,\frac{1-\cos a}{\cos a}\ldots\ldots\ldots\ldots\ldots(1),$$

which gives the required magnitude of the radius.

SECTION III.

FORCES ACTING AT DIFFERENT POINTS RIGIDLY CONNECTED TOGETHER.

23. *If two parallel forces P and Q act at two points A and B rigidly connected, their resultant is a force parallel to either of them; and when P and Q act in the same direction, as in fig.* 1, *it also acts in the same direction and equals $P+Q$:*

fig. 1.

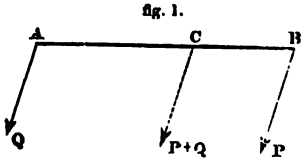
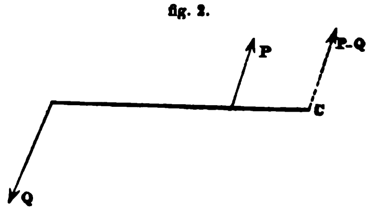


fig. 2.

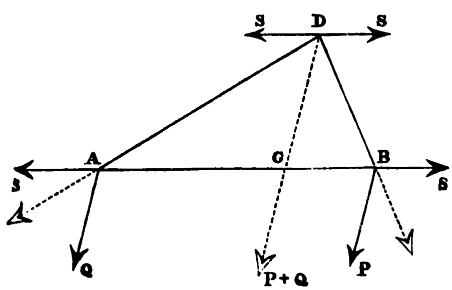


when P and Q act in opposite directions, as in fig. 2, *it acts in the direction of the greater, as P, and equals $P-Q$: in both cases its line of action cuts the line joining AB in a point C such that $AC : BC :: P : Q$.*

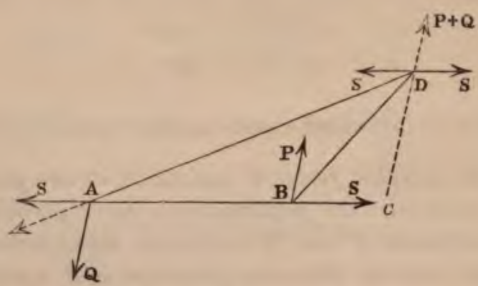
If the convention with regard to the signs of forces explained in Art. (13) be extended to the case of parallel forces, the resultant will always be the *algebraical sum* of the component forces.

24. To prove this proposition.

At A and B respectively apply the two forces S, both acting



in the line AB, but in opposite directions; it is manifest that



the statical effect of the whole system is not altered, for these forces counteract each other.

Let the pairs of forces Q and S acting at A, and P and S at B, be replaced by their respective resultants. In the first figure, when P and Q act in the same direction, it is immediately manifest that the direction of these resultants meet in some point. In the second figure, where P and Q act in opposite directions, if we suppose P greater than Q, the direction of the resultant of P and S will clearly lie nearer to that of P than the direction of the resultant of Q and S does to that of Q; and therefore as P and Q are parallel, in this case also the directions of the resultants must meet. Let, then, D be the point at which they meet, and let their points of application be transferred to D: also let them be here replaced by their original resolved parts. We shall thus have acting at D, the pair of forces S in a line parallel to AB, but in opposite directions, and the forces P, Q in a line DC parallel to the original directions. The pair of forces S may, since they counteract each other, be removed; and the remaining forces P and Q, acting in the same direction in the *first* case, will form a resultant $P+Q$; in the *second*, when their directions are opposite, they will produce a force $P-Q$.

The point C may thus be found. Since the triangle BDC has its sides parallel to the directions of the three forces P, S, and the resultant of these two forces which acting together

upon a point, would keep it at rest, we have by the triangle of forces,

$$BC : DC :: S : P.$$

For a similar reason we have from the triangle ADC,

$$DC : AC :: Q : S.$$

Therefore compounding these two proportions, we get

$$BC : AC :: Q : P,$$

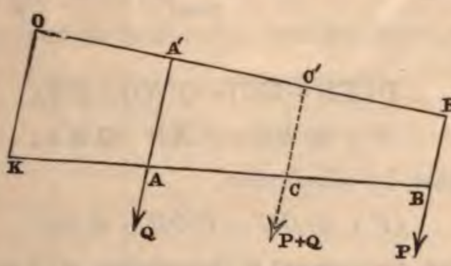
which shews, that the distances of the point C from A and B respectively are inversely proportional to the forces acting at those points.

This fundamental property of parallel forces is more commonly expressed under the following analytical form,

$$P.BC = Q.AC.$$

Def. The moment of a force about a given point is the product of the force into the length of the perpendicular drawn from the given point upon its direction.

24. Let O be any point in the plane of the forces, and let a



line from O perpendicular to the directions of P, Q, and their resultant, cut them in the points A', B', and C', respectively. Draw OK parallel to these directions meeting AB or AB produced in K.

Take the case of P and Q acting in the same direction; and first suppose K to fall upon BA produced: then the result of

the last article may be written

$$P(KB - KC) = Q(KC - KA),$$

or $$(P + Q)\,KC = P.KB + Q.KA.$$

Also, from the similar figures, we have

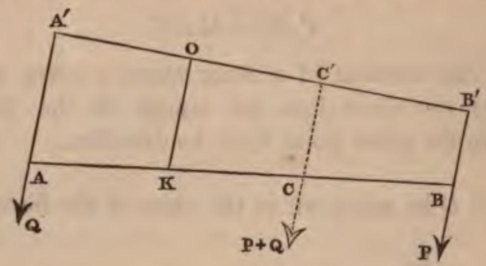
$$\frac{OC'}{KC} = \frac{OB'}{KB} = \frac{OA'}{KA},$$

and therefore, by substitution,

$$(P + Q)\,OC' = P.OB' + Q.OA'.$$

This asserts that the moment of the resultant of P and Q about O is equal to the sum of the moments of P and Q about the same point.

Next, suppose K to fall between A and C; then our form



gives us

$$P(KB - KC) = Q(KC + KA),$$

therefore $$(P + Q)\,KC = P.KB - Q.KA;$$

which becomes, by substitution,

$$(P + Q)\,OC' = P.OB' - Q.OA'.$$

That is to say, the moment of the resultant of P and Q about O is in this case equal to the difference of the moments of P and Q about the same point.

If now, in considering the moment of a force about a given point, we agree to call it positive or negative according as, when we look along the perpendicular *towards* the force, the *direction* of the force is towards our right hand or towards our left, (or

vice versâ), we see that in both the above cases the moment of the resultant about O is equal to the *algebraical* sum of the components about the same point.

The same thing might easily be shewn to be true, when P and Q act in opposite directions; and hence we can make this simple enunciation for all cases:

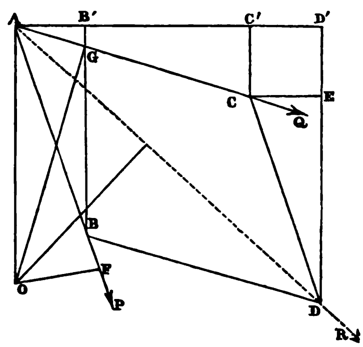
When two parallel forces act upon a body, the moment of their resultant about any point in their plane is equal to the sum of the moments of the forces themselves about the same point.

25. The preceding results may be shewn to be true in an extended form for any system of parallel forces whatever, whose points of application are rigidly connected together: *i.e.*

The resultant of any such system of forces is equal to the algebraical sum of the forces; is parallel to them; and its moment about any point is equal to the algebraical sum of the moments of the several forces about the same point.

26. If two forces, not parallel, act upon a body in one plane, the moment of their resultant about any point in that plane is equal to the algebraical sum of the moments of the respective forces about that point.

Let the two forces be P and Q, and let their directions meet



in A. We may suppose them to be applied there. Take AB, AC to represent them, and complete the parallelogram $ABCD$

upon their directions; then AB, AC, AD, are proportional to P, Q, and their resultant R, respectively.

Let O be any point in the plane of P, Q, and R, and AD' a line perpendicular to OA: draw OF, OG, OH perpendicular to AB, AC, and AD, respectively; also BB', CC', DD' perpendicular to OD', and CE parallel to AD'.

The triangles AFO, AGO, AHO, are evidently similar to the triangles ABB', ACC', and ADD', respectively; therefore

$$\frac{AB'}{AB} = \frac{OF}{OA},$$

or $$AB' = \frac{OF.AB}{OA}.$$

Similarly $$AC' = \frac{OG.AC}{OA},$$

$$AD' = \frac{OH.AD}{OA}.$$

Now, from equal triangles,

$$AB' = EC = C'D',$$

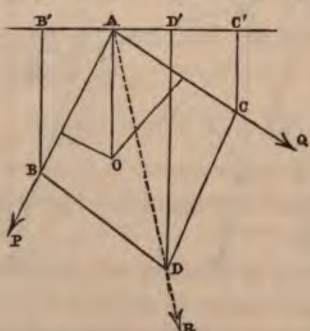
therefore $$AB' + AC' = AD'.$$

Therefore, by substitution,

$$OF.AB + OG.AC = OH.AD,$$

or $$OF.P + OG.Q = OH.R.$$

We have here supposed O to be *without* the angle BAC:



in the annexed figure it falls *within*; we then have

$$AC' - AB' = AD',$$

therefore $$OG.Q - OF.P = OH.R.$$

But in this case, the moment of P about O is by our rule negative; therefore, generally, our proposition holds.

27. It will also be true of R, and any other force whose direction it meets; and, even by the last proposition, of R and any force which is parallel to it. We may thus, by taking each force in succession, prove it to hold for any system of forces whatever in one plane; that is, *for any such system of forces whatever* the moment of the resultant about any point in the plane is equal to the sum of the moments of the respective forces about the same point.

We may look upon this as a general law which holds for the combination of forces, and which is dependent for its proof upon the truth of the proposition of the parallelogram of forces.

28. The assertion of the last Article affords a great step towards the solution of many problems; for from them it follows that in a system of forces of which either all are parallel or those which are not parallel are in the same plane with the resultant of those which are so, the sum of the moments of the forces about any point in the *line of action* of their resultant must equal zero.

29. Also, more generally, if any such system of forces be in equilibrium, the sum of their moments about any point whatever must vanish.

For let $P_1, P_2 \ldots . P_{n-1}, P_n$ be these forces, R the resultant of $P_1 P_2 \ldots . P_{n-1}$: suppose $p_1 p_2 \ldots . p_n$ and r to be the perpendiculars from a given point upon the forces and R respectively; then, by (25) and (27),

$$P_1 p_1 + P_2 p_2 + \ldots . + P_{n-1} p_{n-1} = Rr \ldots \ldots (1).$$

Now by Art. (12) P_n must equal $-R$ and be in the same line with it; therefore $r = p_n$, and $Rr = -P_n p_n$, therefore equa-

tion (1) becomes

$$P_1p_1 + P_2p_2 + \ldots\ldots + P_np_n = 0,$$

which proves the proposition.

30. If one point of a body be fixed, the reactions at this point must be equal and opposite to the resultant of all the other forces acting upon the body, when there is equilibrium; therefore this resultant must pass through the fixed point: hence, by (28), *the sum of the moments of the forces about this point must equal zero.*

31. If two points of a body, or any points in the same straight line, be fixed, the reactions at those points must, when there is equilibrium, counteract all the other forces acting upon the body; it is clear also that they are of such a nature as to counteract all tendency to motion excepting to that about the fixed line as axis. Now all the forces, whatever they be, that are applied to the body, may, as far as their tendency to produce motion about such an axis is concerned, be supposed to be applied at points in one plane perpendicular to the axis, similarly situated, as to this axis, with their actual points of application.

This plane has one point, the point where the fixed axis meets it, fixed: hence, by the last article, the resultant of the applied forces must pass through it, and therefore the sum of their moments about it must vanish.

32. When a body rests with one or more points in contact with a smooth plane, the reactions at those points upon the body are perpendicular to the plane, and their direction is away from it (Art. 9, δ): therefore the resultant of the other forces acting upon the body, since it must be equal and opposite to the resultant of these reactions, must satisfy two conditions:

First, its direction must be towards the plane and perpendicular to it.

Secondly, its point of application must be the same as that of the resultant of the reactions: therefore

(α) When the body touches in one point only, it must ***pass through that point.***

(β) When the body touches in two or more points lying in the same straight line, the reactions at these points are all parallel and in the same direction; therefore the point of application of their resultant lies somewhere between the extreme points of contact, but its position is indeterminate because the reactions themselves may be any whatever. Hence in this case it is sufficient that the resultant of the applied forces pass ***through some point in the line joining the extreme points of section, and between them.***

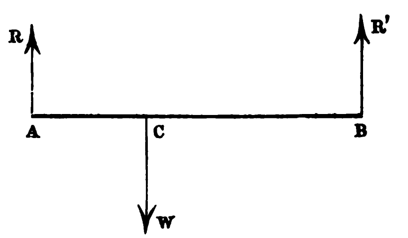
As long as the resultant satisfies these two conditions, its magnitude may be any whatever.

It may be as well to repeat here that all the forces treated of in this section are in one plane, or are capable of being replaced by forces in one plane.

Examples to Section III.

(1). Two persons, A and B, carry a weight of 100 lbs. on a pole between them. The weight being placed two feet from A, and three feet from B, find what portion of it they respectively support.

Let C be the point of the rod at which the weight W is hung, and let R, R' be the forces which A and B respectively



exert at the ends of the pole in a vertical direction: by supposition, these with W acting downwards at C preserve equi-

librium in the system; therefore W is equal and opposite to the resultant of R and R'; or

$$W = R + R',$$

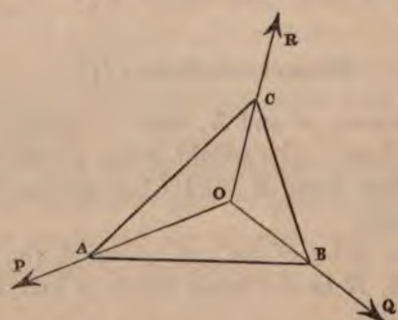
and $$\frac{R}{R'} = \frac{BC}{AC} = \tfrac{3}{2}, \quad \text{(Art. 23)}.$$

Hence we obtain $$R = \tfrac{3}{5}W = 60 \text{ lbs.},$$
$$R' = \tfrac{2}{5}W = 40 \text{ lbs.}$$

(2). A triangle ABC has its point C fixed, and is kept in equilibrium by two forces P and Q acting at A and B respectively, their directions bisecting the angles A and B; shew that if R be the pressure at C,

$$P : Q : R :: \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}.$$

Let the directions of P and Q meet in O; their resultant must be equal and opposite to the pressure at the fixed point



C, because the triangle is in equilibrium; therefore CO is the direction of R, and it bisects the angle C by trigonometry. Because the forces P, Q, R are in equilibrium at O, therefore

$$P : Q : R :: \sin COB : \sin COA : \sin AOB;$$

but $$COB = \pi - \left(\frac{C}{2} + \frac{B}{2}\right),$$

$$= \pi - \frac{\pi - A}{2} = \frac{\pi}{2} + \frac{A}{2};$$

therefore $\sin COB = \cos \dfrac{A}{2}$:

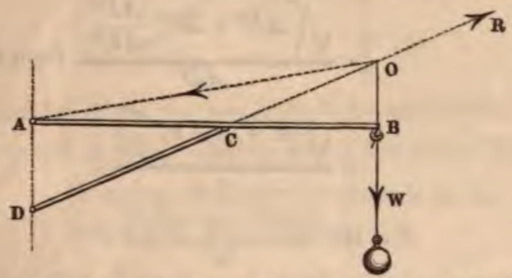
similarly $\sin COA = \cos \dfrac{B}{2}$,

$\sin AOB = \cos \dfrac{C}{2}$;

therefore, by substitution, we get

$$P : Q : R :: \cos \frac{A}{2} : \cos \frac{B}{2} : \cos \frac{C}{2}.$$

(3). AB is a rigid rod without weight, capable of moving freely about a hinge at A; it is kept in a horizontal position by



another rigid rod DC also without weight and having free hinges at C and D. At the extremity B of the rod AB a weight W is hung; supposing A and D to be in the same vertical line, find the action of the hinge A.

The rod AB is kept in equilibrium by the following forces.

The weight W acting vertically downwards at B.

The tension R of the rod DC acting at C in the direction of its length (Art. 9, γ) and the action P of the hinge at A.

The direction of R and W will meet in a point O, P must also pass through this point, therefore AO is its direction.

Let $OAB = \phi$, $AB = a$, $AC = b$, $AD = c$, then

$$\frac{OB}{AB} = \tan \phi,$$

$$\frac{OB}{BC} = \frac{AD}{AC} \text{ ....} (1);$$

therefore $$\frac{BC}{AB} = \frac{AC}{AD} \tan\phi,$$

or $$\tan\phi = \frac{c}{b}\,\frac{a-b}{a} \ldots\ldots\ldots\ldots\ldots (2).$$

Again, the sides of the triangle AOD are parallel and therefore proportional to three forces, which acting at the point O are in equilibrium; therefore

$$\frac{P}{W} = \frac{AO}{AD},$$

$$= \frac{\sqrt{(AB^2 + OB^2)}}{AD},$$

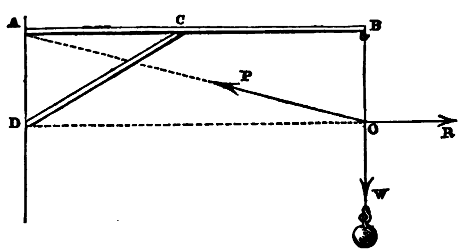
$$= \frac{\sqrt{\left(AB^2 + BC^2\,\frac{AD^2}{AC^2}\right)}}{AD} \text{ from (1)},$$

$$= \frac{\sqrt{\left\{a^2 + (a-b)^2\,\frac{c^2}{b^2}\right\}}}{c};$$

or $$P = W\,\frac{\sqrt{\{a^2b^2 + (a-b)^2c^2\}}}{bc}.$$

(4). If DC were rigidly connected with AB at C, its tension would no longer be in the direction of its length. If besides it were merely in contact with a wall at D, instead of being fastened to it by a hinge, the reaction there would be perpendicular to the wall.

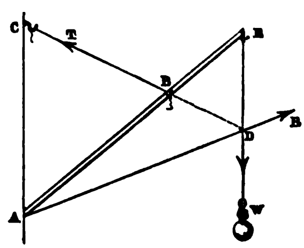
The accompanying figure represents the directions of the



forces in this case, which may be as easily solved as the above.

(5). If a weight be suspended from one extremity of a rod moveable about the other extremity A, which remains fixed, and a string of given length be attached to any point B in the rod, and also to a fixed point C above A in the same vertical line with it, then the tension of the string varies inversely as the distance AB.

Let E be the end of the rod from which the weight W is



hung, and let T be the tension of the string CB.

The system is in equilibrium under the action of the forces

W acting vertically downwards at E,

T acting at B in direction BC,

and the reaction of hinge at A which may be termed R.

Since this last must be equal and opposite to the resultant of T and W acting at D where they meet, AD must be its direction.

Hence the triangle ADC has its sides parallel and therefore proportional to the three forces, which acting at D keep the whole system in equilibrium, therefore

$$T : W :: CD : AC;$$

also from similar triangles CBA, EBD,

$$CD : AE :: CB : AB,$$

therefore
$$T = W\frac{CD}{AC},$$

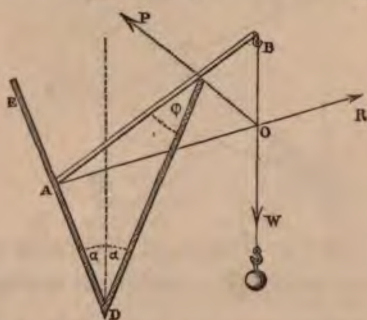
$$= \frac{W}{AC}\,\frac{AE.CB}{AB},$$

$$= W\frac{AE.CB}{AC}\cdot\frac{1}{AB},$$

$$\propto \frac{1}{AB},$$

because W, AE, BC, and AC are all constant.

(6). CDE is a conical shell, having its axis vertical; AB is a rigid rod without weight lying over the edge C, and having



its extremity A in contact with the smooth side DE of the shell, to the extremity B a weight W is hung; required the position of equilibrium of the rod AB.

Let the vertical angle of the conical shell be 2α, the length of AB equal a, and that of CD equal b; also let the angle ACD be represented by ϕ; when the value of ϕ is known, the position required is known.

The beam AB is in equilibrium under the action of the vertical force W acting at B, the reaction P of the edge of the shell acting perpendicular to the beam at C, and the reaction of the side ED of the shell acting perpendicular to that side at A.

Now the directions of any two of these, as of W and R, must meet in some point O, therefore the third force P which is equal and opposite to the resultant of these two must have its direction also passing through O, *i.e.* the line joining C and O must be perpendicular to AB. This consideration enables us to solve the problem.

From this

$$\frac{AC}{AO} = \cos CAO;$$

also
$$\frac{AO}{AB} = \frac{\sin ABO}{\sin AOB};$$

therefore multiplying,

$$\frac{AC}{AB} = \cos CAO \frac{\sin ABO}{\sin AOB},$$

$$= \sin(2a + \phi) \frac{\sin(a + \phi)}{\cos a} \ldots\ldots.(1):$$

also
$$\frac{AC}{CD} = \frac{\sin 2a}{\sin(2a + \phi)} \ldots\ldots\ldots\ldots.(2),$$

therefore dividing (1) by (2),

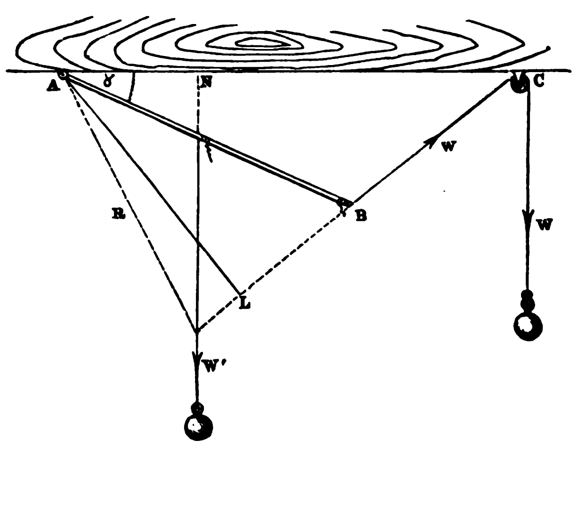
$$\frac{CD}{AB} = \frac{\sin^2(2a + \phi)\sin(a + \phi)}{\sin 2a \cos a};$$

or
$$b \sin 2a \cos a = a \sin^2(2a + \phi) \sin(a + \phi),$$

a cubic equation which will determine ϕ.

P and *R* may be found by the aid of a triangle described upon the three lines *OR, OP, OW*.

(7). A rod without weight *AB* is capable of turning freely about one extremity *A*; to the other *B* is attached a string



which passes over a pully C, and is then attached to a weight W; also at a point G in the rod is fixed a string carrying a weight W'; the points A and C are in the same horizontal line, required the position in which the rod would rest.

Let $BAC = \theta$, $AC = b$, $AB = a$, $AG = c$.

It is clear that the rod is in equilibrium under the action of the following forces:

W the tension of string acting at B in direction BC,

W' acting vertically downwards at G,

and the reaction of the hinge R, suppose at A.

Now the directions of W and W' will manifestly meet in some point O, therefore R which must counteract their resultant must also act through O, that is, its direction must be OA; therefore the sines of the angles AOC, AOW' must be proportional to W' and W respectively: as these angles are necessarily known when θ is known, this proportion would give a relation between θ and constants which would be sufficient to find θ; but the geometrical difficulties of finding this relation are so great, that it is better to take a different course for solving the problem.

Since the resultant of W and W' must when the rod is in equilibrium pass through A, the sums of their moments taken about A must vanish (Art. 28), and therefore producing $W'G$ to meet AC in N, and drawing AL perpendicular to BC,

$$AN.W' - AL.W = 0 \ldots\ldots\ldots\ldots (1).$$

Now $$AN = AG \cos\theta,$$

$$= C \cos\theta,$$

and $$AL.BC = AB.AC.\sin\theta,$$

therefore $$AL = \frac{ab \sin\theta}{\sqrt{(a^2 + b^2 - 2ab\cos\theta)}};$$

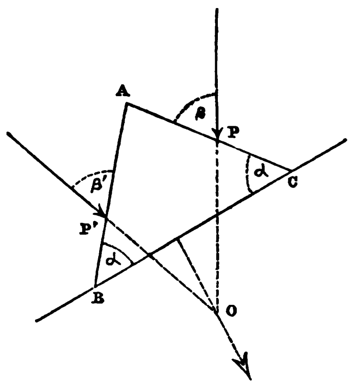
therefore substituting in (1),

$$W'c\cos\theta - W\frac{ab\sin\theta}{\sqrt{(a^2 + b^2 - 2ab\cos\theta)}} = 0.$$

An equation from which θ may be found, it is evidently a cubic equation in $\cos\theta$.

(8). A triangle ABC without weight is placed with its base BC upon a smooth plane.

Two forces P and P' act upon its sides AC and AB re-



spectively, making angles β, β' with them, find the relations between P and P' when there is equilibrium.

Let $$ABC = \alpha', \quad ACB = \alpha,$$

then the triangle is at rest, under the action of the two forces P, P' and the resistances of the plane at every point of BC. Now these resistances are all perpendicular to BC, because the plane is smooth (Art. 9, δ); therefore their resultant is perpendicular to BC, and its point of application is somewhere between B, C; therefore the resultant of P and P' which must counteract it must be perpendicular to BC, and must meet the plane between B and C.

Let P and P' meet in O, and let OR be the direction of their resultant meeting BC in N. Then N will always necessarily lie between B and C if P and P' both act *towards* BC; and ON will be perpendicular to BC if

$$NOP = \alpha + \beta - \frac{\pi}{2},$$

$$NOP' = \alpha' + \beta' - \frac{\pi}{2}.$$

But in this case, since

$$\frac{P}{P'} = \frac{\sin P'OR}{\sin POR}, \quad \text{(Art. 19)},$$

$$= \frac{\sin NOP'}{\sin NOP};$$

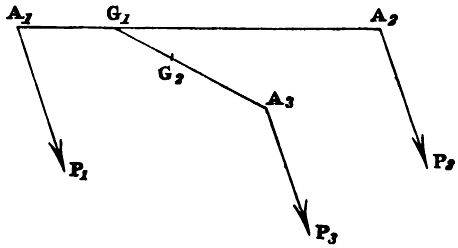
therefore
$$\frac{P}{P'} = \frac{\cos(\alpha' + \beta')}{\cos(\alpha + \beta)},$$

which is evidently the relation required.

SECTION IV.

GRAVITY.

33. Let $P_1 P_2 \ldots P_n$ be any number of parallel forces acting



at points $A_1 A_2 \ldots A_n$, which are rigidly connected together. Join $A_1 A_2$, and in this line take G_1 such that

$$A_1 G_1 : A_2 G_1 :: P_2 : P_1 \ldots\ldots\ldots\ldots (1),$$

then, by Art. (23), the resultant of P_1 and P_2 is equal to $P_1 + P_2$, and passes through G_1 in a direction parallel to P_1 or P_2.

Now join $G_1 A_3$, and take G_2 such that

$$G_1 G_2 : A_3 G_2 : P_3 : P_1 + P_2 \ldots\ldots\ldots\ldots (2),$$

then the resultant of P_1, P_2 and P_3 equals $P_1 + P_2 + P_3$ is parallel to them and passes through G_2: it is manifest from the form (2) that the position of G_2 is independent of the *directions* of the parallel forces.

By proceeding in a similar manner it might be shewn that the resultant of all the forces P_1, $P_2 \ldots P_n$ is equal to the sum of them, is parallel to them, and always passes through a point whose position with reference to $A_1 A_2 \ldots A_n$ is independent of the directions of the forces.

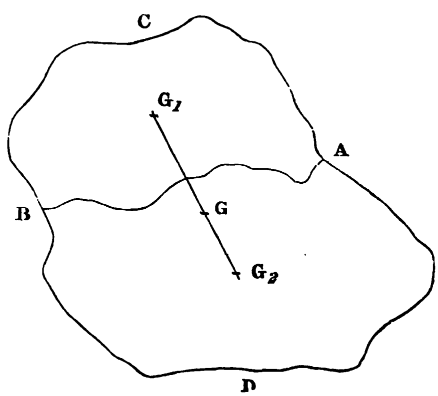
Now the weight of a body is the force which is brought into action by the attraction of the earth upon each of its

particles: but this attraction produces a system of vertical, and consequently parallel forces, whose points of application are the several particles of the body; therefore the weight of a body is equivalent to the resultant of this system; *i.e.* it is a force equal to the sum of these forces, is parallel to them, and passes through a point in the body whose position in the body is invariable, whatever be the direction of the separate forces with reference to any fixed line in the body; or in other words, whatever be the position of the body. This point is called the *Centre of Gravity* of the body.

34. It is evident from this definition that there is but one such point for any given body.

35. If the centre of gravity of two portions which constitute the whole of a body be known, the position of the centre of gravity of the body itself can be easily found.

Thus, suppose the body to be made up of any two parts, as



ABC, ABD: let G_1 be the centre of gravity of the first, W_1 its weight; and G_2 the centre of gravity, and W_2 the weight, of the second.

Then the action of gravity upon ABC is equivalent to the application of a vertical force W_1 to the point G_1: similarly, the action of gravity upon ABD is equivalent to a force W_2 acting vertically at G_2.

Join G_1G_2, and in this line take G such that

$$G_1G : G_2G :: W_2 : W_1;$$

then, by the principles of parallel forces, W_1 at G_1 and W_2 at G_2 may be replaced by a force $W_1 + W_2$ at G; *i.e.* the action of gravity upon the whole body is equivalent to that of a force equal to $W_1 + W_2$ at G; therefore G is by definition the centre of gravity of the whole body.

It is evident that if any two of the points G_1, G_2 be given, the other may be obtained.

36. And generally, if a body can be divided into any number of parts, whose centres of gravity are known, the action of gravity upon the whole body is equivalent to the application of the weights of the several parts at their respective centres of gravity; and hence the centre of gravity of the whole can be easily found by the aid of the propositions that have been given with regard to systems of parallel forces.

If the centres of gravity of the different parts, into which we have supposed a solid divided, all lie in one straight line, it will follow from Art. (23) that the centre of gravity of the whole body will also lie in that line: this is a fact of considerable practical importance, as will be observed in some of the examples appended to this section.

37. If a body or system of heavy particles be in equilibrium under the action of gravity and the resistances of surfaces or fixed points only, it is clear that the resultant of these resistances must be vertical and must pass through the centre of gravity of the body or system: if these two conditions be satisfied, the magnitude of this resultant need not be cared for; it is the nature of forces of resistance to be exerted to a sufficient intensity for producing equilibrium, and not to a greater. This consideration affords an easy means of solution to many of the simpler cases of equilibrium.

Thus, if a body be standing upon a smooth horizontal plane having one or more points in the same straight line, in contact

with the plane, the resistances at these points are vertical, and all act in the same direction; hence their resultant will be vertical, and will pass through some point of this line which lies between the two extreme points of contact, but which is *à priori* indeterminate, because the resistances at the several points are so: if however the direction of the weight of the body, *i.e.* the vertical line through the centre of gravity of the body, meet this line, just so much force will be called forth at each point of contact as will make the resultant of their reactions coincide with this vertical line, and equilibrium will subsist.

But if the direction of the weight of the body pass outside this line, then it cannot be counteracted by the resultant of the resistances which, as was above mentioned, must meet the line, and therefore the body will fall.

Fig. 1.

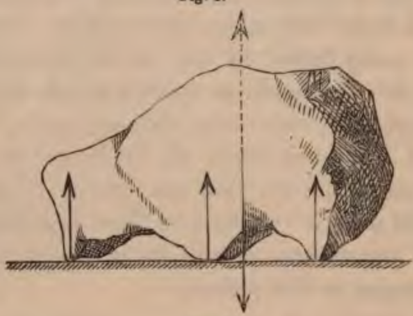
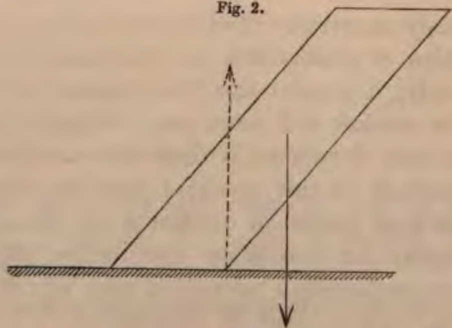


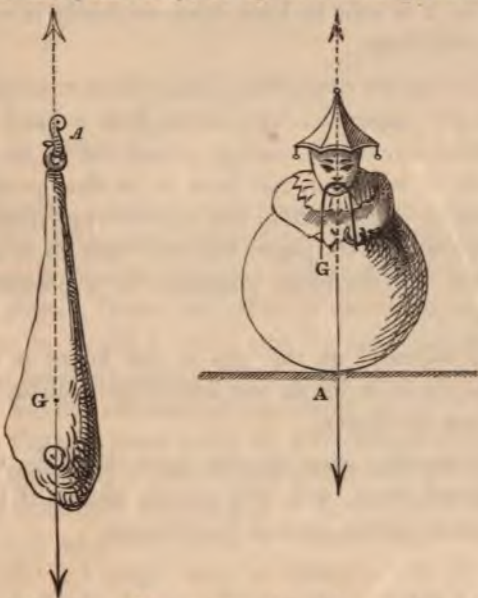
Fig. 2.



Figs. (1) and (2) represent cases where the direction of the weight of the body falls within and without respectively the

line in which the points of contact lie; equilibrium will subsist in the first but not in the second case.

If the equilibrium be produced by the body's being suspended from one point only, or by its touching a surface in one



point only, the preceding reasoning shews that this point of suspension or contact, as A, must be in the same vertical line with the centre of gravity of the body G.

This suggests a method for practically finding the centre of gravity of a body in many cases. Let the body be suspended successively by two separate points in its surface, and when in each case the body has found its position of equilibrium, let the vertical in it which passes through the point of suspension be observed; the point in which the two vertical lines intersect is the centre of gravity required.

38. Def. A body is said to rest in *Stable Equilibrium* when upon being disturbed in a *very slight* degree from its position of equilibrium, it will, upon the disturbing cause being withdrawn, return to its first position.

If, however, it should under such circumstances move farther from its first position, it is said to have been in *Unstable Equilibrium.*

If, again, it remains in the position where the disturbance has placed it, it is said to have been originally in a position of *Neutral Equilibrium.*

The following are examples of these three conditions:

Any weight suspended by a string from a fixed point is in stable equilibrium, for if slightly pulled out of its position, it will, directly it is let go, fall back to its first position. Also a marble resting at the bottom of a smooth cup, if moved in the slightest degree from its place, will roll back to it again when left free, and was therefore originally in a position of stable equilibrium.

A pencil balanced on the end of the finger is in unstable equilibrium, for if it be in the least degree disturbed, it will fall away from the finger.

A penny standing upon its edge upon the table, when rolled through a small space, will still remain at rest if left free; it was therefore at first in neutral equilibrium.

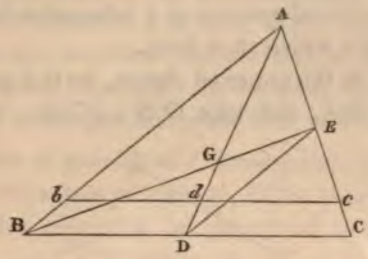
Examples to Section IV.

(1) To find the centre of gravity of a uniform straight line.

The line may be conceived to be made up of pairs of equal heavy particles, each particle of any pair being equidistant from the middle point of the line; therefore the resultant of gravity upon any one pair will pass through this middle point; as this is true of each pair, the resultant of gravity upon all the particles will of course be the sum of these resultants, and will act at the same place; therefore the middle point is the centre of gravity of the whole line.

(2) To find the centre of gravity of a triangular plate.

Let ABC be the triangle; bisect any side BC in D, and join AD.



The triangle, supposed very thin, may be conceived to be made up of a series of heavy straight lines parallel to BC; each one of them is bisected by AD; its centre of gravity is consequently there. Hence the effect of gravity upon the triangle is the same as would be produced by applying the weight of each of the lines parallel to BC at the point where it meets the line AD; but the resultant of such a set of forces would necessarily pass through some point of AD; therefore the centre of gravity of the triangle must be somewhere in AD.

Again, draw BE bisecting AC in E; then, by the same reasoning as that given above, we should find that the centre of gravity of the triangle must be somewhere in BE; but it has been seen that it is also in AD, therefore it must be the point G where these two lines meet.

Draw the line DE; it will be parallel to BA (*Euc.* VI. 2), and therefore the triangles AGB, EGD are similar; hence

$$AG : GD :: AB : ED$$
$$:: 2 : 1,$$

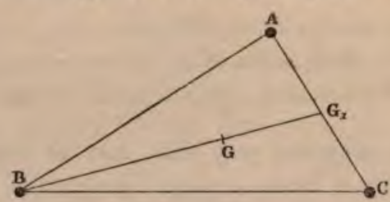
or $$AG = 2GD,$$

and therefore $$GD = \tfrac{1}{3}AD.$$

We thus get an easy rule for finding the centre of gravity of any triangle. Join one of its angles with the point of bisection of the opposite side; the point in this line, whose distance from the bisection of the side is $\frac{1}{3}$ of the whole line, is the centre of gravity required.

(3). If three equal heavy particles be rigidly connected together so as to form a triangle, their centre of gravity coincides with the centre of gravity of a triangular lamina equal in area to the triangle which they form.

Let A, B, C, in the annexed figure, be the particles; bisect AC in G_1, join BG_1, and take $G_1G = \frac{1}{3}G_1B$. G_1 is evidently



the centre of gravity of the particles A and C, and their resultant will equal twice the weight of either of them. Also because

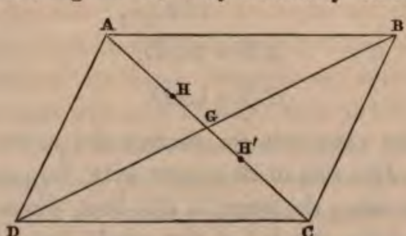
$$G_1G : GB :: 1 : 2,$$

G is the centre of gravity of weight of B at B, and twice as great a weight at G_1; it is therefore the centre of gravity of the three equal particles.

But the construction for finding the position of G has been the same as that for finding the centre of gravity of a triangular lamina whose area is ABC. Therefore these two points are coincident.

(4). The centre of gravity of a parallelogram is at the point of intersection of its diagonals.

For let $ABCD$ be any parallelogram; draw the diagonals AC, BD intersecting in G, they mutually bisect each other;

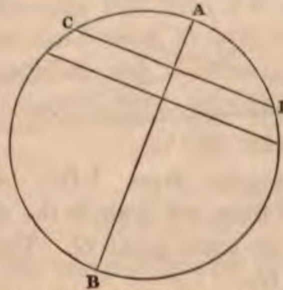


therefore the centre of gravity H of the triangle DAB is in

the line AG at a distance $= \frac{1}{3}GA$ from G; and the centre of gravity H' of the triangle DCB is at the same distance $= \frac{1}{3}GC$ from G: also the triangles are both equal, and consequently their weights are so, therefore the resultant of the two weights is half-way between H and H'; or in other words, the centre of gravity of the whole parallelogram is at G.

(5). The centre of gravity of a circular area is at its centre.

Let AB be a diameter of a circle, CD a chord perpendicular to it; the circular area may be conceived to be made up of



heavy lines, all parallel to CD; each of these will be evidently bisected by AB, and will therefore have its centre of gravity in AB; hence by Art. (36) the centre of gravity of the whole circle must be in AB.

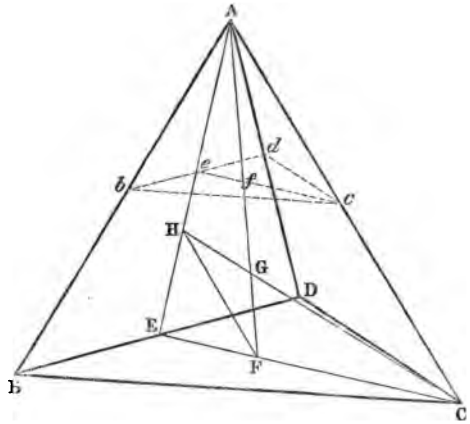
Similarly it might be shewn that it must be in any other given diameter; it is therefore at the point where all the diameters intersect, *i.e.* at the centre of the circle.

(6). Also the centre of gravity of a circular uniform ring is at its centre. For the ring may be conceived to be composed of pairs of equal heavy points, each pair being the extremities of a diameter: now the centre of gravity of every such pair is at the bisection of the diameter, *i.e.* at the centre of the ring; hence the centre of gravity of the aggregate of these pairs, or of the complete ring, must also be there.

(7). To find the centre of gravity of a pyramid of uniform density, having a triangular base.

Let $ABCD$ be any such pyramid. Bisect the edge BD in E; join AE, CE; in CE take EF equal $\frac{1}{3}EC$, and in AE

take EH equal $\frac{1}{3}EA$; F and H are evidently the centres of



gravity of the triangular faces BDC, ABD respectively; join AF, CH; these lines are both in the plane of AEC, and will therefore meet in some point G. The centre of gravity of the pyramid is at G.

For we may consider the pyramid to be made up of heavy lamina parallel to the face BDC; let bdc be one of these; because the plane ABD meets the parallel planes BDC, bdc in the lines BD, bd, these lines are parallel; similarly DC, dc are parallel, and also CB, cb. Let the plane AEC meet the plane bdc in the line ec; this must, for the reason just given, be parallel to EC; let also AF meet ec in f.

Because bd is parallel to BD, therefore

$$be : BE :: Ae : AE,$$
$$:: ed : ED;$$

now $BE = ED$, therefore $be = ed$.

Similarly, since the lines ec, EC are parallel, we should get

$$ef : fc :: EF : FC,$$
$$:: 1 : 2;$$

therefore ce is drawn from the angle c to the bisection of the opposite side bd of the triangle bdc, and f is a point in ce, such that $ef = \frac{1}{3}ec$; f is therefore the centre of gravity of the triangle bdc.

It might be shewn in the same manner that the centre of gravity of every lamina parallel to BDC is in the line AF; therefore the centre of gravity of the pyramid itself must be in it. (Art. 36.)

Similarly the centre of gravity of the pyramid must be in the line CH, because it could be shewn that the centre of gravity of every lamina parallel to ABD is in CH.

Hence G, the point where these two lines meet, must be the centre of gravity.

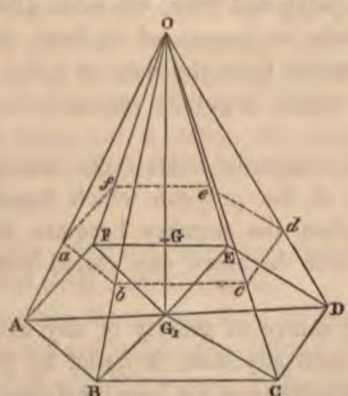
Join FH: since this line cuts AE and CE in the same proportion, it must be parallel to AC, therefore the triangles AGC, HGF are similar, and

$$\begin{aligned} AG : GF &:: AC : HF, \\ &:: CE : EF, \\ &:: 3 : 1; \end{aligned}$$

therefore $GF = \frac{1}{4} AF.$

Hence, to find the centre of gravity of a triangular pyramid, we must draw a line from one of its solid angles to the centre of gravity of its opposite side, and take a point in this line at a distance from the base of $\frac{1}{4}$ the whole line; this point will be the centre of gravity required.

(8). Let $OABCDEF$ be a pyramid whose base is the polygon



$ABCDEF$. Let G_1 the centre of gravity of the base be found;

this can be done by considering it to be formed of as many triangles as it has sides. Join OG_1. In OG_1 take G such that $G_1G = \frac{1}{4} G_1O$, and through G draw a plane *abcdef* parallel to $ABCDEF$; this plane manifestly cuts all lines drawn from O to the base in the same proportion that it does OG_1; the centre of gravity of the triangular pyramid ABG_1O will therefore be in it, for the line from O to the centre of gravity of its base ABG_1 will be divided by the plane into two parts which are to each other as is 1 to 3: similarly the centres of gravity of each of the other triangular pyramids BCG_1O, CDG_1O, &c. will all be in the plane *abcdef*; therefore the centre of gravity of the aggregate of these pyramids will also be in that plane.

But the aggregate of these pyramids is the whole pyramid $OABCDEF$; hence the centre of gravity of the whole pyramid must be in this plane.

Again, this pyramid may be supposed to consist of a series of lamina parallel and similar to $ABCDEF$, and it may be shewn that the centre of gravity of each of them is in the line OG_1; hence the centre of gravity of the pyramid itself must also be in that line, but it has been proved that it lies in the plane *abcdef*, it is therefore at the point G when this plane meets OG_1.

Hence, generally, to find the centre of gravity of any pyramid upon a polygonal base, we must draw a line from its vertex to the centre of gravity of its base, and take point in this line at a distance from the base of $\frac{1}{4}$ the whole line; this point will be the centre of gravity required.

(9). This rule evidently holds quite independently of the number of sides of the polygon which forms the base, it is therefore true when the number becomes indefinitely great, *i.e.* when the base becomes any closed curve, as a circle, ellipse, &c.

Therefore the centre of gravity of any uniform cone, right or oblique, and upon any base, is found by joining the vertex with the centre of gravity of the base, and then taking a point in this line at a distance from the base equal to $\frac{1}{4}$ of the whole line.

(10). The centre of gravity of the surface of a right cone is in the axis of the cone at a distance from the base of $\frac{1}{3}$ of the whole axis.

For the surface may be supposed to be made up of an infinite number of very small equal isosceles triangles having their vertices in the vertex of the cone, and their bases in the circumference of the base of the cone; each of them will have its centre of gravity at a point in its length, whose distance from the bottom of the triangle equals $\frac{1}{3}$ of its whole length, or $\frac{1}{3}$ of the side of the cone. Now the aggregate of these centres of gravity will evidently form a ring in the surface of the cone at a distance of $\frac{1}{3}$ of the side from the base; we may therefore replace the surface of the cone, as far as the effect of gravity upon it is concerned, by this ring; but the centre of gravity of this ring is at its centre, that is, at the point where its plane cuts the axis of the cone, which is clearly at the distance from the base of $\frac{1}{3}$ of the whole axis.

Therefore the centre of gravity of the surface of a right cone is in the axis at a distance from the base equal to $\frac{1}{3}$ of the whole axis.

(11). The centre of gravity of a uniform sphere is at its centre. For the sphere may be supposed to be made up of a series of parallel circular plates perpendicular to a given diameter. As the plates are uniform, each will have its centre of gravity in this diameter because its centre is there; therefore the sphere itself, which is made up of these, will also have its centre of gravity in this diameter. But the same may be said of any other diameter; therefore the actual position of the centre of gravity is at the point where the diameters all meet, that is, at the centre.

(12). The centre of gravity of the surface of a sphere or of a uniform spherical shell is at its centre.

For the surface may be supposed to be made up of a series of circular rings, whose planes are parallel to each other and perpendicular to a given diameter; the centre of each of these rings, which is also its centre of gravity because the rings

are all uniform, is therefore in this diameter; hence, by the same reasoning as in the last case, the point where the diameters intersect, that is the centre of the spherical shell, is its centre of gravity.

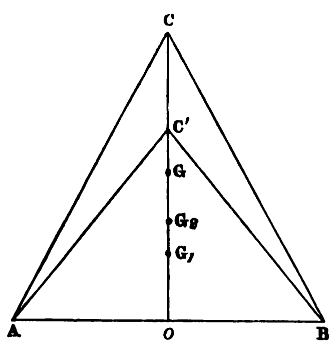
(13). Triangles are described in a given circle upon a given chord as base; their centres of gravity all lie upon the circumference of a similar arc of a circle.

If from the middle point of the chord a line be drawn to the vertex of any one of the triangles, a point one-third the way up this line is the centre of gravity of the triangle. If from this point lines be drawn parallel to the sides of the triangle, it can be easily seen that they will always cut the chord in the same two points; and thus a triangle will have been formed which will always have the same vertical angle and the same base; its vertex will therefore trace out a circle. Q. E. D.

(14). Two isosceles triangles, ABC, ABC', are described upon the same base AB and on the same side of it; find the centre of gravity of the area $ACBC'$ between the triangles.

Draw the line $CC'O$ which bisects the base AB in O.

Let G_1, G_2, G, be the centres of gravity of the triangles ABC', ABC, and the area $ACBC'$ respectively; if then W_1



be the weight of triangle ABC', W_2 that of ABC, since ABC is made up of the two parts ABC' and $ACBC'$, therefore W_2 acting at G_2 is equivalent to W_1 acting at G_1, together with $W_2 - W_1$ (the weight of $ACBC'$) acting at G.

Hence we have by Art. (24),

$$OG_1 . W_1 + OG.(W_2 - W_1) = OG_2 . W_2:$$

but if $OC' = h_1$, $OC = h_2$, then $OG_1 = \frac{1}{3} h_1$, $OG_2 = \frac{1}{3} h_2$,

therefore $\quad \frac{1}{3} h_1 . W_1 + OG(W_2 - W_1) = \frac{1}{3} h_2 . W_2;$

therefore $$OG = \frac{1}{3} \frac{h_2 W_2 - h_1 W_1}{W_2 - W_1}.$$

Now the weights of the triangles are proportional to their areas; therefore

$$W_1 : W_2 :: h_1 : h_2,$$

therefore $$OG = \frac{1}{3}(h_2 + h_1).$$

(15). A circular hole of given size is cut in a square board as near as possible to one corner; find the centre of gravity of the board afterwards.

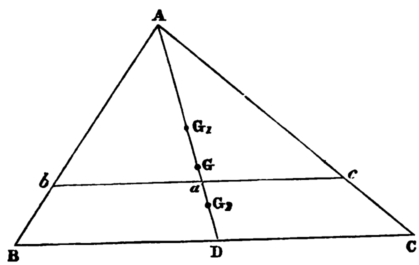
The required centre of gravity necessarily lies in the line produced, which joins the centre of the hole cut out, and the middle point of the square; its position must be found by the consideration that the weight of the board which is left after the circle has been cut out applied at it, together with the weight of the circle applied at its centre, is equivalent to the weight of the whole board applied at its centre. Of course the weights of the different parts are proportional to their areas.

(16). A triangle is cut off from a parallelogram by a right line which bisects two adjacent sides; find the centre of gravity of the remainder.

It is clear that the base of this triangle is bisected by that one of the diagonals of the parallelogram which passes through its vertex; hence the centre of gravity of the triangle lies in this diagonal, as does also that of the whole parallelogram; and therefore the centre of gravity of the part required must be in it: its position may be found by the method employed in the preceding examples.

(17). A triangle is bisected by a line parallel to its base, find the centre of gravity of its four-sided half.

Let bc be the line parallel to the base which bisects the triangle ABC. Draw AdD bisecting bc and BC, then the



centres of gravity of the triangles Abc, ABC, and therefore that of the part $BdcC$, are in the line AdD; let G_1, G, and G_2 be these points respectively.

Then the weight of the whole triangle ABC at G is equivalent to the weight of the triangle Abc at G_1, together with the weight of the figure $BbcC$ at G_2; but since these two last figures are by supposition equal, their weights are equal, and therefore

$$GG_1 = GG_2,$$

or

$$AG - AG_1 = AG_2 - AG \ldots\ldots\ldots\ldots (1).$$

Now $\quad AG = \frac{2}{3}AD$, and $AG_1 = \frac{2}{3}Ad$,

therefore, by substitution in (1),

$$\tfrac{2}{3}(AD - Ad) = AG_2 - \tfrac{2}{3}AD,$$

or

$$AG_2 = \tfrac{2}{3}(2AD - Ad) \ldots\ldots\ldots\ldots (2);$$

also, since the triangle ABC is double of triangle Abc, and they are similar, we have

$$AD^2 : Ad^2 :: 2 : 1,$$

therefore

$$Ad = \frac{1}{\sqrt{2}} AD;$$

therefore, substituting in (2),

$$AG_2 = \tfrac{2}{3}\left(2 - \frac{1}{\sqrt{2}}\right) AD,$$

$$= \frac{\sqrt{2}\,(2\sqrt{2} - 1)}{3} AD.$$

From this we should get

$$DG_2 = \frac{\sqrt{2}}{3} Dd,$$

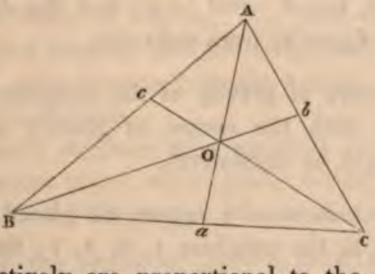
a result which might also be obtained directly from (1).

(18). ABC is a triangle, $\alpha\beta\gamma$ are the middle points of the sides opposite to A, B, and C respectively; $\alpha\beta$ and $\alpha\gamma$ are joined, and the figure $A\beta\alpha\gamma$ is removed from the triangle: determine the centre of gravity of the remainder.

Join $A\alpha$. The portion removed is evidently a parallelogram equal to half the triangle; and since $A\alpha$ is one of its diagonals its centre of gravity is at the middle point of $A\alpha$. Also the centre of gravity of the whole triangle is in $A\alpha$ at a distance $\frac{1}{3}A\alpha$ from α. The centre of gravity of the remaining part must therefore be in $A\alpha$ too; it is moreover at the same distance from the centre of gravity of the triangle as the centre of gravity of the part removed is; hence it can easily be found that its distance from α equals $\frac{1}{6}$ of $A\alpha$.

(19). Three particles are placed at the angles of a triangle, whose weights are proportional to the opposite sides; shew that their centre of gravity coincides with the centre of the inscribed circle.

Let ABC be the triangle, then the weights applied at



A, B, C respectively are proportional to the sides BC, CA, and AB.

Draw the lines Aa, Bb, Cc bisecting the angles A, B, and C; these meet in some point O, which is the centre of the inscribed circle.

Because Aa bisects the angle A, it cuts the base BC into segments, such that

$$Ba : Ca :: BA : CA, \text{ (Euc. Bk. vi.)},$$
$$:: \text{weight at } C : \text{weight at } B;$$

therefore a is the centre of gravity of the two heavy particles at B and C, *i.e.* the three particles at their respective angles are equivalent to the particles at B and C placed together at a, whilst the other particle remains at A. Hence the centre of gravity of all the three particles must be in Aa. In a similar manner, it could be shewn that it is in Bb; and therefore its actual position must be at O.

(20). If the sides of the triangle ABC be bisected in the points DEF, then the centre of the circle described within the triangle DEF is the centre of gravity of the perimeter of the triangle ABC, considered as three uniform heavy rods.

For the weight of each side may be supposed to be collected at its middle point; it is also proportional to the side, and therefore proportional to the opposite side of the similar triangle DEF: hence the problem becomes the same as that just solved.

(21). The sides of a triangle considered as three uniform heavy rods are 3, 4, and 5 feet long; find the distance of their common centre of gravity from each side.

(22). The centre of gravity of the periphery of a triangle cannot coincide with the centre of gravity of the triangle unless the triangle be equilateral.

(23). At the corners of a square are placed weights which are proportional to the numbers 1, 3, 5, 7; find their centre of gravity.

It will be found convenient to obtain first the centre of gravity of the pair 1 and 7, and also of the pair 3 and 5; the centre of gravity required manifestly bisects the line joining these two points.

(24). Four rings are placed at equal distances upon a straight rod without weight; each successive ring is twice as heavy as the preceding one. Determine the position of that point in the rod about which the whole system will balance.

The point required is the centre of gravity of the four heavy rings; for since the resultant of their weights acts there, if this point be supported, the whole system will be in equilibrium.

Its position may be easily found by the consideration that the sums of the moments of the weights, taken about it must vanish. (Art. 28.)

(25). Given the position of the centre of gravity of an uniform triangular lamina, deduce that of a rod in which the weight of any particle is proportional to its distance from one end.

The triangular lamina will become such a rod, if the lines parallel to its base, of which it may be considered to be formed, be each condensed into its middle point; but the centre of gravity of the triangle lies in the line which bisects all the lines parallel to the base, and its position is independent of the breadth of the base; hence it will remain unaltered when the triangle is in its condensed state. It is then the centre of gravity required of the proposed rod.

(26). A triangle may be supposed to be made up of rods parallel to its base; in a given case these rods are uniform throughout their length, but the weight of the same length of each varies as its distance from the vertex. Find the centre of gravity of the triangle.

This is merely the case of a pyramid upon a square base, condensed into one plane, which passes through the vertex, and cuts the base in a line parallel to one of its sides.

(27). Given the centre of gravity of a circular arc; find that of the area of a sector of a circle.

(28). A triangular plate hangs by three parallel vertical threads attached to its corners, and supports a heavy particle placed upon it. Prove that if the threads are of equal strength,

a heavier particle may be supported at the point which would be the centre of gravity of the plate, were it heavy, than at any other point of the plate.

Suppose the greatest possible heavy particle, which the strings will support at the centre of gravity of the plate, to be placed there. Since it is in equilibrium, its weight must be equal and opposite to the resultant of the tensions of the three strings; but these are vertical, and at the angles of the triangle; hence, since their resultant passes through the centre of gravity of the triangle, they must be equal (Ex. 3); therefore each equals $\frac{1}{3}$ of the weight of the particle.

If the particle be placed in any other position upon the plate, and there remain in equilibrium, the resultant of the tensions of the strings, *i.e.* the sum of them, must still be equal and opposite to its weight: but in this case the tensions cannot all be equal, as their resultant does not pass through the centre of gravity of the plate; therefore one, at any rate, must be greater than $\frac{1}{3}$ of the weight of the particle; but this is by supposition the greatest weight that any one of them can bear; hence equilibrium cannot subsist with the particle in this position, or a greater weight can be supported at the centre of gravity of the plate than anywhere else.

(29). A rectangular block of wood, $ABCD$, is placed with its side CED on a horizontal plane, E being some point in CD; find the largest portion BCE which can be cut off without the remainder $ABED$ falling over.

The centre of gravity of the whole block is known, and also that of the triangle BCE when E is known, hence the centre of gravity of the remainder $ABED$ can be found; call this G. BCE is the greatest possible, consistently with the terms of the question, when G is vertically above E. (Art. 37.)

(30). An inclined plane makes an angle of 10° with the horizon; how many sides has the regular polygon which will just trundle down the plane? (The plane of the polygon being vertical.)

The vertical line through the centre of gravity of the polygon must clearly just pass through the lower point of the side which is in contact with the plane, but the centre of gravity of the polygon is manifestly the centre of the figure; therefore in this polygon the line joining the centre of the figure with one of the angles, must make with the adjacent side an angle which is the complement of 10°. But this angle in a regular polygon must always be the complement of ½ the angle which each side subtends at the centre, *i.e.* of $\frac{180°}{n}$, if n be the number of sides of the polygon. Hence we have here

$$\frac{180°}{n} = 10°,$$

or the number of sides required is 18.

(31). Why does a person carrying a heavy weight in his hand lean towards the opposite side? (Art. 37.)

(32). When a man rises from a chair, why does he bend his body forward and his legs back?

(33). If a sphere were not homogeneous (not made of uniform material) how could it be practically ascertained whether or not its centre of gravity were at the centre of the figure? (Art. 37.)

SECTION V.

FRICTION.

39. When two surfaces, not smooth, are in contact, and it is attempted to make the one move upon the other, a force due to the want of smoothness arises and tends to prevent the motion; this force is *Friction*. It acts upon each surface in the tangent plane common to both surfaces at the point of contact, and in a direction exactly opposite to that in which the other forces tend to make this point move.

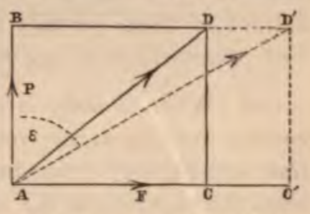
In general, just so much of this force is called into action as will serve to keep the point of contact at rest, but there is a certain limiting value in each case, beyond which it is found that friction cannot be exerted: this limiting value is always proportional to the normal reaction at the point of contact, the proportion depending only upon the nature of the two surfaces.

Thus, if P be the normal force at the point of contact of two surfaces in a supposed case, F the greatest amount of friction that can be exerted at that point with the normal force P, then $F = \mu P$ where μ is a numerical quantity which is constant, so long as the materials of which the two surfaces are formed are the same. This quantity μ is termed the *coefficient of friction* of the particular surface to which it refers, and its value can only be obtained by experiment.

If the surfaces be in contact throughout a plane area, the relations just mentioned will hold between the limiting amount of friction and the normal pressure at each point of it: and hence it can be easily seen that the same must be true for the resultant of the pressures and the resultant of the limiting frictions; the coefficient of friction thus remaining unchanged is independent of the extent of surface in contact.

40. Suppose A to be the point of a given surface which is in contact with another, and let AB, AC represent the normal force P and the friction F respectively acting upon A; they are therefore in the directions of the normal, and tangent to the surfaces at A.

Draw AD the diagonal of the parallelogram described upon AB, AC; then AD represents both in magnitude and direction the resultant of P and F. It is clear that the larger F is for the same value of P, the farther will AD lie from the normal AB, and that its greatest angular distance from AB will correspond to the greatest value of F, *i.e.* to $F = \mu P$: let AC' represent this value of F, and AD' the resultant in this case of P and F; then



$$\tan D'AB = \frac{BD'}{AB} = \frac{AC'}{AB},$$

$$= \frac{\mu P}{P} = \mu;$$

therefore $$D'AB = \tan^{-1}\mu.$$

It is usual to represent this angle by the symbol ϵ. The *magnitude* of the resultant evidently equals $\sqrt{(P^2 + F^2)}$, and may therefore be as great as the circumstances of the case require, for P can always be exerted to any extent.

41. It thus appears that when two surfaces are in contact, their resultant reaction upon each other may be anything whatever as to magnitude, and also as to direction within a certain limiting angle with the normal, which we have called ϵ. This is often by far the most convenient light in which to view the effect of friction. The best way of finding μ for different substances is to observe this angle ϵ; there are many devices for effecting this object.

42. Our definition of *perfect smoothness* (Art. 9, 8) is equivalent to $\mu = 0$, and therefore ϵ equal 0. μ put equal to ∞, and therefore $\epsilon = \frac{\pi}{2}$, gives a case where the one surface could not possibly slide upon the other; the surfaces are then termed *perfectly rough.* Both these may be looked upon as limiting cases of the action of surfaces upon one another, for they neither of them actually occur in nature, although we often find extremely near approximations to them.

43. When a body is in contact with a plane surface, the contact existing at any number of points in the same straight line, and neither surface being supposed to be perfectly smooth, since the reactions at the several points will not be necessarily perpendicular to the plane, their resultant will depend for its direction as well as its magnitude upon the circumstances of the particular case; it may have any inclination to the normal to the plane within a given limit (the ϵ of Art. 40) and be of any magnitude, but it must still pass through some point in the line in which lie the points of contact, and between the extreme points of contact. Hence equilibrium will be always preserved if the resultant of the other forces acting upon the body, whatever be its magnitude, pass through some point in this finite line, and have any direction within the abovenamed limiting angle ϵ.

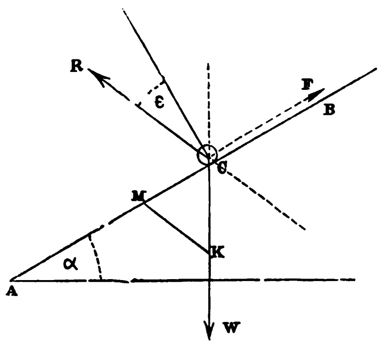
44. In the supposed case of *perfect roughness,* the reaction at each point of contact only differs from that which can be exerted upon a *fixed point,* in that it cannot be *towards the plane,* in other respects it is indeterminate; hence it is only necessary for equilibrium, that the resultant of the other forces acting upon the body should *not act from the plane,* and should pass through some point in the before-mentioned line.

A full explanation of friction, with methods for determining its amount, and tables giving the values of μ and ϵ for a great number of different substances, may be found in the *Third Treatise on Mechanics,* published by the Society for the Diffusion of Useful Knowledge.

Examples to Section V.

(1). A particle of weight W rests upon an inclined plane whose coefficient of friction and inclination are given; the particle is attached by an extensible string to the top of the plane; the tension of the string is always in a constant proportion to its length: find the greatest distance from the top of the plane at which the particle will remain in equilibrium.

Let AB be the plane inclined at an angle α to the horizon, $\tan \epsilon$ its coefficient of friction, x the distance of the



particle from B when the greatest amount of friction is exerted, *i.e.* in the supposed case; F the tension of the string at this length, R the reaction of plane, making angle ϵ with the normal at C.

The particle C is kept in equilibrium by the forces F, R, and W. In CW take any point K, and draw KM parallel to CR and meeting AB in M.

Then the sides of the triangle CKM are parallel to and therefore proportional to the three forces which keep C at rest; we have therefore

$$\frac{F}{W} = \frac{MC}{CK} = \frac{\sin MKC}{\sin KMC} = \frac{\sin(\alpha + \epsilon)}{\cos \epsilon} \quad \ldots\ldots \text{(1)},$$

$$\frac{R}{W} = \frac{MK}{CK} = \frac{\sin MCK}{\sin CMK}$$

$$= \frac{\cos \alpha}{\cos \epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}.$$

Again, if T be the tension of the string at a known length a, we have

$$F : T :: x : a \ldots\ldots\ldots\ldots\ldots (3),$$

$$F = \frac{x}{a} T;$$

therefore, substituting in (1),

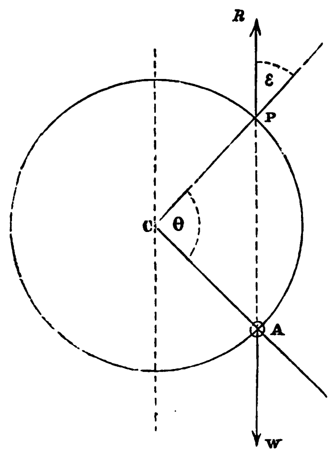
$$\frac{x}{a} T = W \frac{\sin(\alpha + \epsilon)}{\cos \epsilon},$$

$$x = a \frac{W}{T} \frac{\sin(\alpha + \epsilon)}{\cos \epsilon},$$

which gives the distance required.

(2). A heavy particle is attached to the edge of a hoop without weight, which is then hung over a peg. Given the coefficient of friction between the peg and hoop equal $\tan \epsilon$; find the position of equilibrium in which the greatest amount of friction is called forth.

Let A be the particle, C the centre of the hoop, P the peg, and θ the angle between CA and CP when there is equilibrium.



The hoop is kept at rest by the two forces, W the weight of the particle, and the reaction R at P, they must therefore be equal

and their directions must be in the same straight line, *i.e.* R must be vertical and A immediately below P.

Also, since the greatest amount of friction is exerted, R is inclined at angle ϵ to the normal at P; therefore

$$\frac{\pi - \theta}{2} = \epsilon,$$

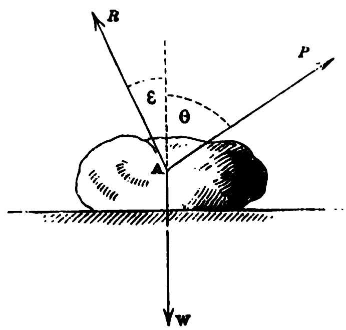
or

$$\theta = \pi - 2\epsilon.$$

Any larger value of θ will make the angle between the vertical and normal at P less than ϵ: hence, by what has been explained of friction, every such larger value of θ will give a position of equilibrium.

(3). Find the direction in which a given force P must act, so that the weight which it can just move along a rough horizontal plane may be the greatest possible.

Let θ be the required inclination of P to the vertical, W the corresponding weight, R the reaction of the horizontal



plane upon the body; then, since the body is only just on the point of moving, and therefore the greatest possible amount of friction is called into action between the body and the plane, R is inclined at an angle $\epsilon = \tan^{-1}\mu$ to the vertical, where μ is the coefficient of friction.

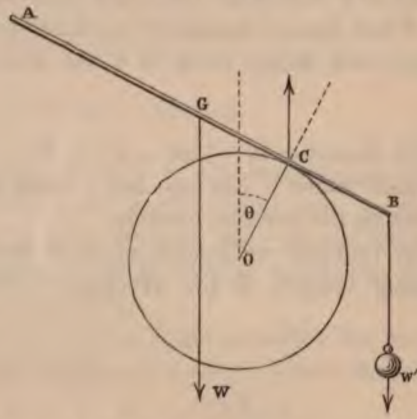
Since these three forces are just in equilibrium, we have

$$\frac{P}{W} = \frac{\sin \epsilon}{\sin(\theta + \epsilon)},$$

or $$\sin(\theta+\epsilon) = \frac{W}{P}\sin\epsilon.$$

As θ increases with W, the greatest value of W is evidently when $\frac{W}{P}\sin\epsilon = 1$, or $W = \frac{P}{\sin\epsilon}$, and $\theta = \frac{\pi}{2} - \epsilon$.

(4). A heavy rod AB rests across a rough horizontal cylinder whose centre is O. The weight of the rod is W, and a



weight W' is attached to one of its extremities B: find the positions of equilibrium of the rod.

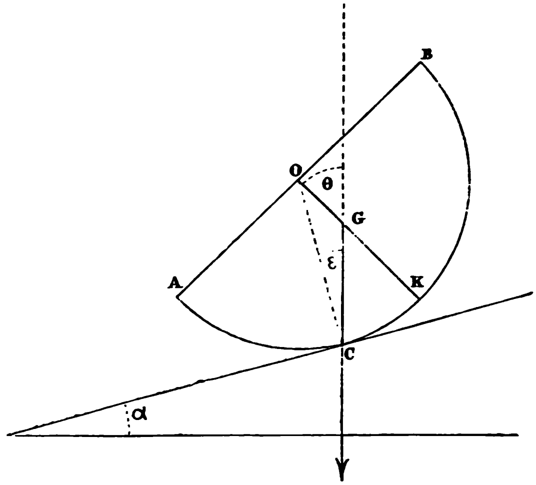
Let the point of contact be C, and let OC make an angle θ with the vertical.

Then the forces acting upon the rod are W at G, the centre of gravity of the rod, and W' at B, both vertically downwards, and the reaction of the sphere at C; hence this reaction must be equal and opposite to the resultant of W and W'.

In the first place, then, the point C in the rod must be such that
$$GC : BC :: W' : W.$$

Next, this point may be placed anywhere upon the sphere provided that θ be not greater than ϵ, because R can make any angle with OC not greater than ϵ; $\tan\epsilon$ being the coefficient of friction between the rod and cylinder.

(5). A hemisphere ACB is placed upon a rough plane which is inclined to the horizon at an angle a: given that the



coefficient of friction between the two surfaces is $\mu = \tan \epsilon$; find the limiting position of equilibrium.

G the centre of gravity of the hemisphere may be taken in the radius OK, which is perpendicular to AB, at a distance OG from $O = h$: let r be the radius of the hemisphere. Let θ represent the angle which OG makes with the vertical in a position of equilibrium.

Since the only forces acting upon the hemisphere are its own weight vertically downwards at G, and the reaction which may be of any magnitude at C, it is sufficient and necessary for equilibrium that this reaction be vertical and pass through G.

In other words, there will always be equilibrium provided that G be vertically above C, and GCO be not greater than ϵ.

But when GC is vertical,

$$GCO = a,$$

and therefore, since $\dfrac{OG}{OC} = \dfrac{\sin GCO}{\sin OGC}$,

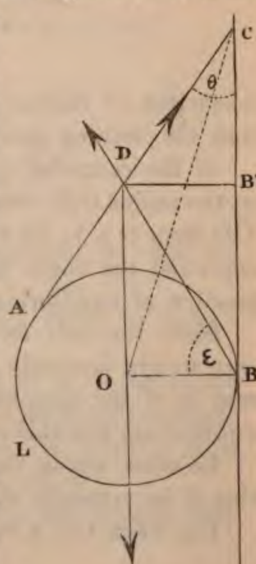
$$\frac{h}{r} = \frac{\sin a}{\sin \theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence we see that generally there is but one position in which G can be vertically above C; if however the least value of θ given by (1) be greater than BGO, its supplement will also evidently correspond to a possible position of the hemisphere such that G shall be vertically above C.

If G be placed vertically above C, it is still necessary for equilibrium that $GCO = \alpha$ be not greater than ε: unless this condition be fulfilled, the hemisphere can in no way rest with its curved surface upon the plane; hence the extreme case for which equilibrium would be possible, would be when $\alpha = \varepsilon$.

(6). A string fastened to a point in a rough wall is wrapped round a ball, which is then allowed to hang against the wall; determine the limiting position of equilibrium.

Let BAL be the ball in its position of equilibrium, O its centre, B the point where it touches the wall, A the point where the string first touches the ball, C the point where it is fixed. The forces acting upon the ball are, the tension of the string AC, the reaction of the wall at B, and the weight of the ball acting vertically downwards at O.



Let the directions of the weight and of the tension meet in D; then, that there may be equilibrium, it is sufficient and necessary that the direction of the reaction at B also pass through D. Now this will always take place, provided that DBO be less than $\varepsilon = \tan^{-1}\mu$, where μ is the coefficient of friction between the ball and the wall, and the limiting case will be when DBO just equals ε. (Art. 41).

Let, in this case, ACB be denoted by θ, join OC, and draw DB' perpendicular to BC; then, because OC bisects the angle ACB, and DOC is equal to OCB, therefore

$$DOC = DCO,$$

and $$DO = DC,$$

hence $$\frac{DB'}{DC} = \frac{DB'}{DO} = \frac{OB}{OD},$$

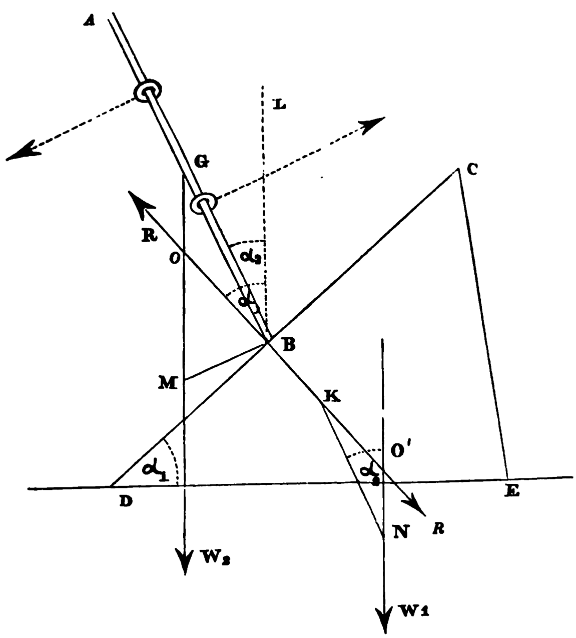
or $$\sin\theta = \cot\epsilon,$$

$$= \frac{1}{\mu},$$

which gives the value of θ required.

It is easily seen from this, that equilibrium could not subsist at all, if μ were less than 1.

(7). A heavy smooth beam AB, whose weight is W_2, passes through two smooth rings, so placed that α_2 is the angle be-



tween the beam and the vertical BL, and rests with its extremity B upon the side of a heavy wedge CDE, whose weight is W_1. If the coefficient of friction between the plane and the ground be μ equal $\tan\alpha_2$, and α_1 be the inclination of

the plane to the horizon, the system will be on the point of moving, provided that

$$W_1 \tan \alpha_2 = W_2 \tan (\alpha_1 - \alpha_2).$$

Let R be the mutual reaction between the beam and wedge, this will act at B perpendicular to DC, because the end of the beam is supposed smooth.

The wedge will be in equilibrium under the action of the following forces: its weight W_1 acting vertically downwards, the reaction R acting at B perpendicular to DC, and the resultant of the reactions of the ground upon its base.

Since the wedge is just on the point of moving, the greatest possible amount of friction must be called into action between the wedge and the ground, and therefore the resultant of their mutual reactions must be inclined at an angle $\tan^{-1}\mu$ (Art. 41), *i.e.* at an angle α_2 to the vertical; hence the resultant of R and W_1, which must counteract this force, must also be inclined at an angle α_2 to the vertical.

Let the directions of R and W_1 meet in O', take $O'N$ to represent W_1, and draw NK making an angle α_2 with the vertical $O'N$, then the sides of the triangle $KO'N$ are parallel, and therefore proportional to R, W_1 and their resultant; hence we have

$$\frac{R}{W_1} = \frac{KO'}{O'N},$$

$$= \frac{\sin \alpha_2}{\sin (\alpha_1 - \alpha_2)} \ldots\ldots\ldots\ldots\ldots (1).$$

Again, the beam is in equilibrium under the action of the following forces: R acting at B perpendicular to DC, W_2 its weight acting vertically downwards, and the reactions of the rings.

Now, because the beam is quite smooth, these reactions must be perpendicular to it; therefore the resultant of R and W_2 which must counteract these reactions must also be perpendicular to the beam.

Let the directions of R and W_2 meet in O, and draw BM perpendicular to AB; then the triangle OBM has its sides

parallel to and therefore proportional to the three forces R, W_2, and their resultant; hence we have

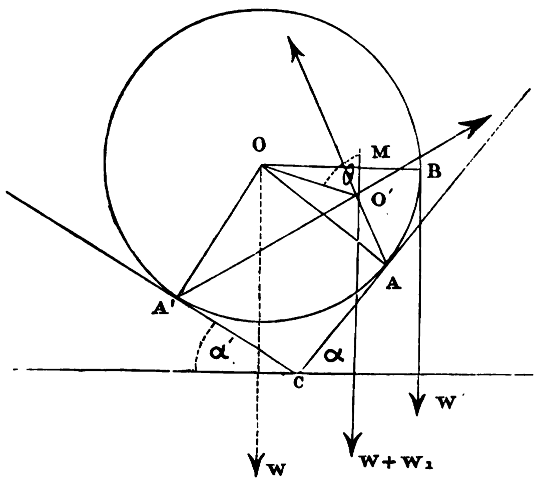
$$\frac{R}{W_2}=\frac{BO}{OM}=\frac{\sin OMB}{\sin OBM},$$

$$=\frac{\cos MGB}{\cos OBG}=\frac{\cos a_2}{\cos (a_1-a_2)}\ldots\ldots(2):$$

therefore eliminating R between (1) and (2), we get

$$W_1 \tan a_2 = W_2 \tan (a_1 - a_2).$$

(8). A cylinder ABA', whose weight is W, rests between two inclined planes CA, CA', the coefficient of friction being



$\tan\lambda$, and the inclination of the planes to the horizon a and a'; a weight W' is suspended from B, the extremity of the horizontal radius OB of the cylinder by a string which is coiled round the cylinder. If W' be the greatest weight which can be suspended consistently with equilibrium, then

$$\frac{W'}{W+W'}=\frac{\sin\lambda\cos\left(\dfrac{a-a'}{2}-\lambda\right)}{\cos\dfrac{a+a'}{2}}.$$

By the nature of the question, the greatest possible amount of friction will be exerted at A and A', and therefore the directions of the reactions at those points will be inclined at angles equal λ, to the radii AO and $A'O$; let these directions meet in O'.

The cylinder is kept in equilibrium by these reactions, the weight W at O and W' at B. But these two weights will have a vertical resultant equal $W+W'$ passing through a point M in OB, such that

$$OM : OB :: W' : W + W' \dots\dots\dots\dots (1):$$

as this must be counteracted by the resultant of the before-mentioned reactions, it must also pass through O'; or in other words, $O'M$ must be vertical, and therefore perpendicular to OB. Join OO', and call $OO'M = \theta$; then if r be radius of the cylinder, the proportion (1) becomes

$$OO' \sin\theta : r :: W' : W + W' \dots\dots\dots (2).$$

Now in triangle $OO'A'$,

$$\frac{OO'}{r} = \frac{\sin OA'O'}{\sin OO'A'} = \frac{\sin\lambda}{\sin(\alpha' + \theta + \lambda)} \dots\dots\dots (3),$$

and from triangle $OO'A$

$$\frac{OO'}{r} = \frac{\sin OAO'}{\sin OO'A} = \frac{\sin\lambda}{\sin(\theta - \alpha + \lambda)} \dots\dots\dots (4);$$

therefore from (3) and (4),

$$\sin(\alpha' + \theta + \lambda) = \sin(\theta - \alpha + \lambda),$$

therefore $$\alpha' + \theta + \lambda = \pi - (\theta - \alpha + \lambda),$$

therefore $$\theta = \frac{\pi}{2} + \frac{\alpha - \alpha'}{2} - \lambda,$$

therefore from (3)

$$OO' = r\frac{\sin\lambda}{\cos\frac{\alpha + \alpha'}{2}}.$$

Substituting in (2) we get

$$\frac{\sin\lambda\cos\left(\dfrac{\alpha-\alpha'}{2}-\lambda\right)}{\cos\dfrac{\alpha+\alpha'}{2}} : 1 :: W' : W' + W.$$

SECTION VI.

THE SIMPLE MECHANICAL POWERS.

45. The object of all machines considered in a statical point of view is to enable a certain force as P, which is called the *Power*, to be in equilibrium with a second force, as W which is termed the *Weight*. If of these counteracting forces, P be smaller than W, the power is said to act at a mechanical advantage; if it be greater than W, at a mechanical disadvantage. These distinctive names are given to the two forces because we are most familiar with the use of machines when employed for the purpose of raising or moving heavy bodies by the application of a small force or power.

46. All machines or mechanical powers may be considered as being formed by combinations of cords and rods with hard surfaces. The simplest of them are the Lever, Wheel and Axle, Toothed Wheels, Pulley, Inclined Plane, Wedge, and Screw.

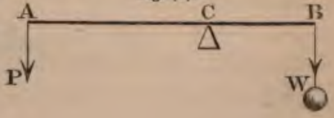
The Lever.

47. Def. The simple lever is a rigid rod without weight, capable of turning freely about some fixed point in its length, which is called the *fulcrum*.

The power and weight are applied at two other points of the rod. The relative positions of these points with respect to the fulcrum suggest the division of levers in three classes as follows:

Suppose C to be the fulcrum, A the point of application of the *power* which may be the tension of a cord produced by pulling by the hand; B that of the *weight*, as of a body; then fig. (1)

Fig. (1).



represents a lever of the first kind where the fulcrum is between the power and the weight. The *Crowbar* in some methods of use is an example of this species; *Scissors* and *Carpenters' Pincers* are double levers of the same kind, the joint being the fulcrum.

Fig. (2) represents a lever of the second kind, where the fulcrum is at one end, the power at the other, and the weight in the middle. A *crowbar* may be used so as to be a lever of this kind. A *cork-squeezer* and an *oar* are other examples; in the last, the blade of the oar in the water is the fulcrum. *Nut-crackers* are double levers of the same kind.

Fig. (2).

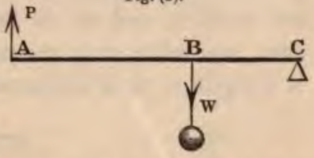
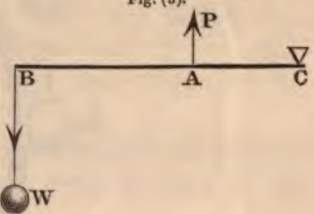


Fig. (3) represents a lever of the third kind, where the fulcrum is again at one end of the rod, but the power is in the middle, and the weight at the other end. The *bones of the arm* where the muscle produces the power, are examples of this kind of lever; *Blacksmiths' tongs, shears,* &c. are double levers of the same sort.

Fig. (3).



48. Def. The distances AC, BC of the points of application of the power and weight from the fulcrum, are termed the arms of the lever; they need not be in the same straight line; when they are not so, the lever is said to be a bent lever.

The directions of the power and weight may be any whatever.

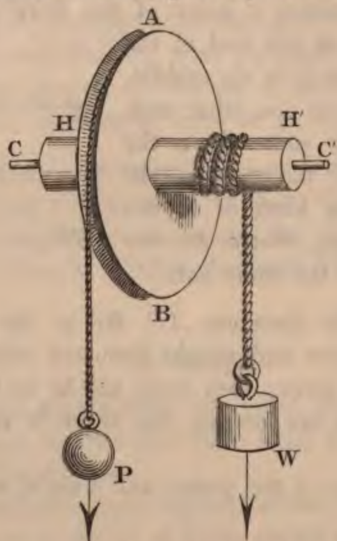
49. In all cases the lever is kept in equilibrium by the forces called the power and weight respectively, and the reaction of the fulcrum; hence the resultant of the power and the weight must equal this reaction of the fulcrum, and its direction must pass through the fulcrum, which is insured (Art. 28) when the algebraical sum of their moments about the

fulcrum vanishes: this condition is sometimes enunciated in the following form—*the power and the weight are inversely proportional to the lengths of the perpendiculars drawn from the fulcrum upon their directions.*

As the reaction of the fulcrum, or its equivalent, the pressure upon the fulcrum, is equal to the resultant of the power and weight, it follows that when the power and weight are parallel, and act in the same direction, it is equal to their sum, and when they are parallel, but act in opposite directions, it is equal to their difference (Art. 23).

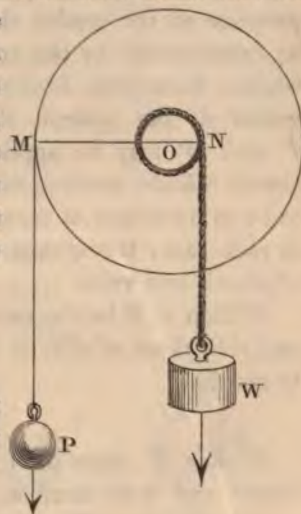
Wheel and Axle.

50. The annexed figure represents a wheel and axle. HH' is a cylinder capable of turning freely about pivots CC' at its



extremities, AB is a wheel firmly fixed to the cylinder HH' and having the same axis. The power P acts by means of a cord or rope which is wrapped round the circumference of the wheel; the weight W is applied in a similar manner by a rope which is wound round HH', but which runs in an opposite direction to that of P.

By the aid of a figure representing a section of the machine, made perpendicular to the axis, it is not difficult to see that the power and weight respectively are always acting at the extremities of a lever of the first kind, whose arms are the radii of the wheel and axle, and fulcrum the axis of the machine; hence, for equilibrium, the moments of these forces about the fulcrum must be equal; or if R be radius of the wheel, r of the axle, we must have

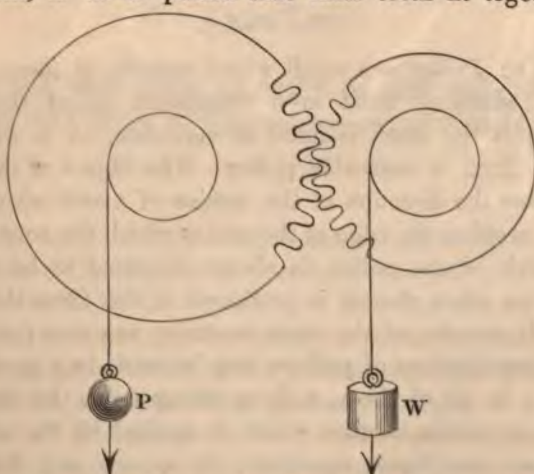


$$\frac{P}{W} = \frac{r}{R}.$$

The common apparatus for raising a bucket out of a well is a familiar example of this machine.

Toothed Wheels.

51. Suppose two wheels, whose circumferences are indented with teeth, to be so placed that their teeth fit together, and



their axes lie parallel to each other: if then one of them be

made to revolve about its axis by a force applied to it, it will communicate motion to the other by means of the mutual pressure of the teeth: this tendency to produce motion may be counteracted by the action of a force applied to the second wheel; these two forces may be called the *power* and the *weight* of the system, the P and the W as in the lever. P and W may be applied to their respective wheels in an almost infinite number of ways; it is not uncommon for them to be so by means of a rope coiled round an axle to the wheel in each case; the system then becomes a combination of a pair of wheels and axles.

If then r, R be the radii of P's wheel and axle respectively, and $r'R'$ those of Q's, it is not difficult, by the aid of Art. 50, to see that

$$\frac{P}{W} = \frac{R}{R'} \times \frac{r'}{r}.$$

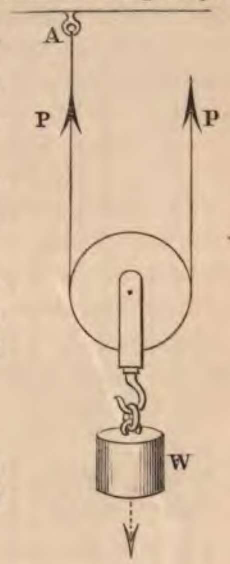
If $R = R'$, since the teeth are set at equal distances on each wheel, and their numbers on each are therefore inversely proportional to the radius, we observe that the above form becomes

$$\frac{P}{W} = \frac{\text{number of teeth on } P\text{'s wheel}}{\text{number of teeth on } W\text{'s wheel}}.$$

The Pulley.

52. The *Pulley* is a small wheel capable of turning about its axis, which is fixed in a framework called the block; according as the block is fixed or moveable, so is the pulley termed a fixed or moveable pulley. The object of the pulley is to change the direction of the tension of a cord which passes over it: as either the edge of the pulley which the cord touches, or the axis of the pulley, is always supposed to be perfectly smooth, no other change is produced in the force thus transmitted; it remains of the same intensity at every point of the cord. Combinations of pulleys may be made in a great variety of ways; in all of them, a force acting upon the first string is made to sustain another which is applied by the aid of the last; these two forces constitute the power and the weight P and W.

53. The annexed figure represents a single moveable pulley, by means of which a force P applied to one end of the cord supports a weight W which is attached to the block; the cord passes freely round the pulley, and is attached to a fixed point A.



Since every thing is supposed to be smooth, the tension of the string will be the same throughout its length.

If the cords be parallel, the sum of their tensions must be equal to the weight which they counteract; and therefore neglecting the weight of the pulley,

$$2P = W.$$

Generally, the weight of the pulley is too considerable to be omitted, put it equal w; then

$$2P = W + w.$$

This relation may be put into the following convenient shape,

$$P - w = \tfrac{1}{2}(W - w).$$

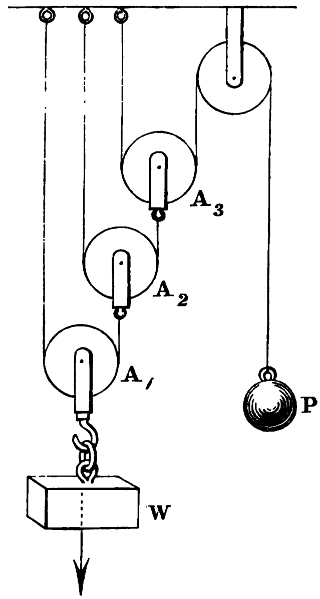
54. If the cords make an angle $2a$ with each other, we should find the relation between the power and weight to be

$$P \cos a - w = \tfrac{1}{2}(W - w).$$

55. The annexed figure represents a system of n pulleys in which the string that passes round any one pulley has one of its ends fastened to a fixed point, and the other attached to the block of the next pulley; the weight is applied to the block of the lowest pulley, and the power to the extremity of the first string. This is called the *First System of Pulleys.*

Suppose all the strings to be parallel; and represent the tension of the string which passes round the lowest pulley A_1 by T_1, that of the string passing round A_2 by T_2, and so on:

then, calling the weights of the pulleys w_1, w_2...w_n, respectively,



we have for the equilibrium of the lowest pulley, by the preceding Article,

$$2T_1 = (W + w_1) \ldots\ldots\ldots\ldots (1).$$

Similarly, since the tension of the string round A_1, when applied to the block of A_2, takes the place of W in the preceding case,

$$2T_2 = T_1 + w_2 \ldots\ldots\ldots\ldots (2).$$

Similarly &c. = &c.

$$2T_{n-1} = T_{n-2} + w_{n-1} \ldots\ldots\ldots\ldots (n-1),$$

$$2T_n = T_{n-1} + w_n \ldots\ldots\ldots\ldots (n).$$

Now, multiplying equation (2) by 2, equation (3) by 2^2, &c...... and equation (n) by 2^{n-1}, and adding them altogether, we get

$$2^n T_n = W + w_1 + 2w_2 + \&c. \ldots + 2^{n-1} w_n.$$

Also it is evident that $T_n = P$: therefore the relation required is

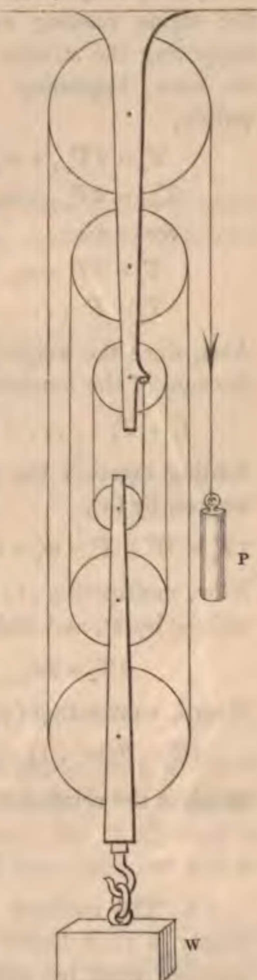
$$2^n P = W + w_1 + 2w_2 + \ldots + 2^{n-1} w_n.$$

If the weights of the pulleys be all equal w, this becomes

$$2^n P = W + w(2^n - 1),$$

or

$$P - w = \frac{1}{2^n}(W - w).$$



56. In the *Second System of Pulleys* the *same* cord passes round all the pulleys of which each alternate one is fixed in one immoveable block, the others are fixed in another block to which the weight is attached.

If the pulleys be so arranged that the several parts of the string passing from pulley to pulley are all parallel, since the tensions of these parts must all be the same and each equal P, the lower block and the weight W attached to it are held up by n parallel forces each equal P, where n is twice the number of the pulleys at the lower block, and therefore equal to the whole number of the pulleys employed. If, then, we represent the weight of the lower block together with the weights of its $\frac{n}{2}$ pulleys by w, we have

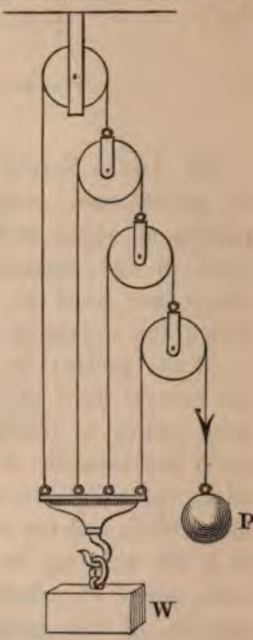
$$nP = W + w,$$

which is the relation existing in this system between P and W.

57. In the *Third System* of pulleys, the cord which passes over any one has one extremity fastened to the block to which the weight is attached, while the other is attached to the block of the next pulley: the highest pulley is fixed.

Let A_1 be the lowest pulley, A_2 the next higher, and so on, A_n being the highest and fixed one; $w_1, w_2 \ldots w_n$ the weights

of these pulleys respectively, and T_1, $T_2 \dots T_n$ the tensions of the string passing over them: then, supposing the strings to be all parallel, we have, beginning with the highest pulley,



$$T_n = 2T_{n-1} + w_{n-1} \dots\dots\dots (1),$$
$$T_{n-1} = 2T_{n-2} + w_{n-2} \dots\dots\dots (2),$$
$$\&c. = \&c.$$
$$T_2 = 2T_1 + w_1 \dots\dots\dots (n-1),$$
$$T_1 = P \dots\dots\dots\dots\dots (n).$$

Also, since the weight W is held up by the sum of the tensions

$$T_1 + T_2 + \dots + T_n = W \dots (a).$$

Adding together the first n equations, we get, by (a),

$$2T_n = W + P + w_1 + w_2 \dots + w_{n-1} \dots (\beta).$$

Now, multiplying (1) by 2, (2) by 2^2... and (n) by 2^n, and adding, we have

$$2T_n = 2w_{n-1} + 2^2 w_{n-2} + \dots 2^{n-1} w_1 + 2^n P \dots\dots (\gamma).$$

Hence, subtracting (γ) from (β),

$$W = P(2^n - 1) + w_1(2^{n-1} - 1) \dots + w_{n-2}(2^2 - 1) + w_{n-1},$$

which is the relation required between P and W.

The Inclined Plane.

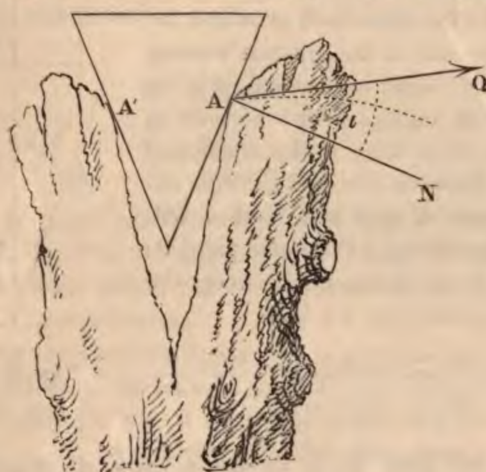
58. The inclined plane, as a mechanical power, is of use simply to raise heavy bodies. When considered statically, the question must be, *what power* acting in a given direction will keep the proposed weight in equilibrium upon the plane; or in *what direction* must a given force act in order to effect the same thing? In either shape the question is one of the simplest that can be propounded in Statics, and has already appeared in the foregoing sections; its solution may be generally stated thus:

If a be the inclination of the plane to the horizon, ε that of the power P to the plane, W the weight,

$$\frac{W}{P} = \frac{\cos \varepsilon}{\sin \alpha}.$$

The Wedge.

59. The wedge is a solid triangular prism, formed of some hard material, such as steel: its section made perpendicular

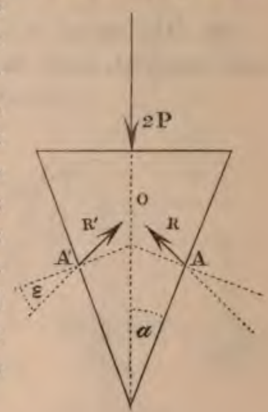


to its length is generally an isosceles triangle. Its use is to separate the parts of a body or system which have, from any cause, a tendency to approach each other; this is effected by introducing its edge between them and then applying a force to its back sufficient for the purpose.

Suppose A and A' to be two particles of a body forcibly held separate by means of the wedge; if the wedge were removed, the two parts of the body would collapse; and on account of the rigidity of the system, A and A' would begin to approach each other by moving along some definite curved line or path; hence the effective force of the wedge which keeps A and A' apart must act along this curve, *i.e.* its direction must coincide with the tangent to the curve at A.

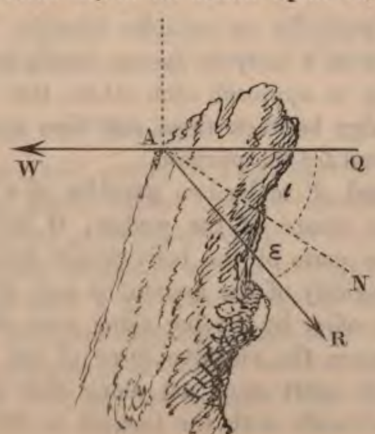
Suppose AQ to be this direction, making an angle i with the normal to the wedge at A: call the force applied to the back of the prism $2P$, this effective force along AQ, W; these are the power and weight respectively of this mechanical system.

To find the relations between them, we may suppose A and A' to be symmetrically situated with regard to the wedge, and the force $2P$ to be applied at the centre of the back and perpendicularly to it: the reactions between the wedge and obstacle will be the same both at A and A', equal R say; and if the greatest amount of friction be called into action, will be inclined at an angle ε to the normals at A and A', where $\tan\varepsilon$ is the coefficient of friction between the wedge and obstacle. Hence R at A and R at A' will meet the direction of $2P$ at some point O: and since these three forces keep the wedge in equilibrium, we have



$$\frac{2P}{R} = \frac{\sin\{\pi - 2(\alpha + \varepsilon)\}}{\sin\left\{\frac{\pi}{2} - (\alpha + \varepsilon)\right\}} = \frac{\sin 2(\alpha + \varepsilon)}{\cos(\alpha + \varepsilon)} = 2\sin(\alpha + \varepsilon).$$

Again, this force R, in its action upon A, counteracts the



force which tends to make A approach A', and which we have called W, together with the other forces of molecular action which the application of the wedge may have called forth, such as the resistance of the ground transmitted by the rigidity of the substance to A, or any other cause which renders the body fixed and capable of receiving the application of the wedge. These are necessarily perpendicular to AQ, for, by supposition, W alone gives the tendency to motion; hence the resultant of R and W, which would counteract these, must be perpendicular to AQ, and therefore we have, by the triangle of forces,

$$\frac{R}{W} = \frac{1}{\cos(\iota + \epsilon)}.$$

Hence
$$\frac{P}{W} = \frac{\sin(\alpha + \epsilon)}{\cos(\iota + \epsilon)}.$$

If there be more particles than A in contact with the side of the prism, the P in the above equation must be that portion of the whole power which may be supposed to be appropriated to keeping A in equilibrium.

The wedge is but little known as a machine for producing *equilibrium* between forces: in its different forms of hatchet, knife, chisel, plough, &c., it is used to enable the force applied to its back or its equivalent, whether impulsive or not, to produce a dynamical effect; *i. e.* to force apart two or more points of a rigid body, to drive them through a space away from each other, and not to keep them remaining at rest, separated by an interval of space. It is only at the termination of each *stroke* of the instrument that equilibrium of the kind supposed in the problem really does subsist. In the interval between the strokes, if the instrument remains in its place, friction alone counteracts the effort of the rigid body to force the wedge out again.

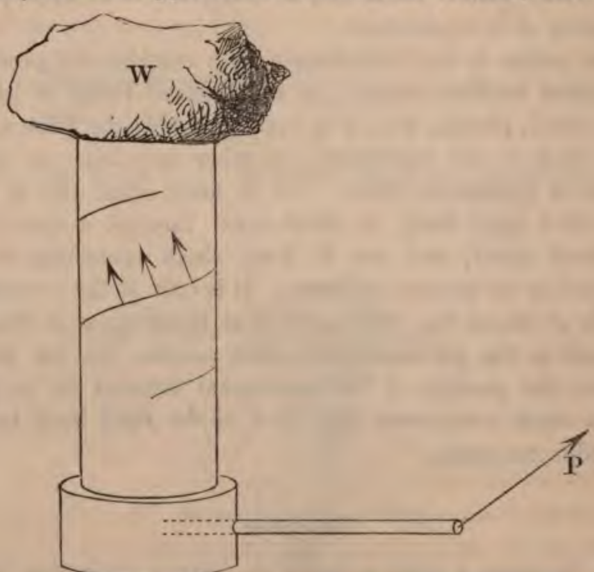
The Screw.

60. Suppose a solid cylinder to have a projecting rib or thread wrapped round its surface in such a way as always

to make the same angle with the axis; suppose also a hollow cylinder of the same radius, and having a groove winding round its inside surface in exactly the same way as the thread does round the exterior of the solid cylinder: if now the first of these be placed so as to fit into the second and be turned about its axis, the thread will slide along the groove, and the solid cylinder will advance within the other: such a contrivance is called a *Screw*. The interior cylinder and its shell are sometimes distinguished by the terms *Male and Female Screw*. Sometimes too the female screw is called the nut.

It is evident that if the nut be fixed, and the screw be made to revolve within it, its extremity may be pressed against any obstacle, and thus act so as to keep in equilibrium some other given force; this force, and that which gives the screw the tendency to revolve, may respectively be called the *weight* and *power* of the system, W and P as before.

In considering the relation between P and W, we may for simplicity's sake suppose the screw to be vertical, and to be



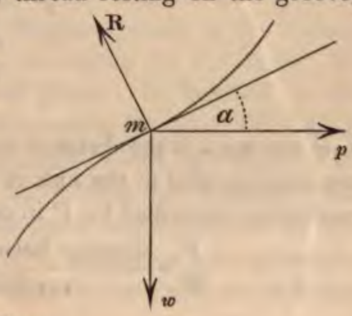
used for supporting a heavy body whose weight is W, the

power P being horizontal and tending to turn the cylinder round by acting at the extremity of an arm perpendicular to the axis of screw and length a.

The whole system will be in equilibrium under the action of the force W acting downwards, the force P acting horizontally, and the resistance of the groove upon the thread perpendicular to the thread at every point.

We may conceive this as equivalent to each particle of the thread bearing its own share of the weight W, and being acted upon by a horizontal force capable of producing the effect upon it which P actually does : thus,

Let m be any particle of the thread resting on the groove, w the part of W supported by it, p the horizontal force equivalent to a portion of P, R the normal reaction of the groove upon m, a the constant angle which the line touching the thread at m makes with the horizon; then, since these three must be in equilibrium,



$$\frac{p}{w} = \frac{\sin Rmw}{\sin Rmp}$$

$$= \frac{\sin a}{\cos a}$$

$$= \tan a.$$

Similarly, if p_1w_1, $p_2w_2 \ldots p_nw_n$ correspond to m_1, $m_2, \ldots m_n$, all the other points of the thread

$$\tan a = \frac{p}{w} = \frac{p_1}{w_1} = \frac{p_2}{w_2} = \ldots\ldots = \frac{p_n}{w_n} = \frac{p + p_1 + \ldots + p_n}{w + w_1 + \ldots + w_n},$$

by a common theorem.

Now the sums of the w's must make up the whole weight W; and all the p's acting horizontally at the surface of the cylinder, (*i.e.* at a distance $= r$ the radius of the screw from its axis), form, with the force P acting at distance a which they counter-

act, a many-armed lever, which is in equilibrium about its fulcrum the axis of the cylinder: hence the sums of the moments of these p's must equal that of P about the axis of the screw. These two considerations give us

$$w + w_1 + w_2 \ldots\ldots + w_n = W,$$

$$pr + p_1 r + \ldots\ldots + p_n r = Pa,$$

or

$$p + p_1 + \ldots\ldots + p_n = P\frac{a}{r}.$$

Hence, substituting above,

$$\frac{P}{W}\frac{a}{r} = \tan\alpha,$$

$$\frac{P}{W} = \frac{r}{a}\tan\alpha,$$

$$= \frac{2\pi r\tan\alpha}{2\pi a}.$$

Now $2\pi r\tan\alpha$ is the distance between the threads of the screw measured parallel to the axis of the screw, and $2\pi a$ is the circumference described by P in one revolution; therefore

$$\frac{P}{W} = \frac{\text{distance between the threads}}{\text{revolution of } P}.$$

Balances.

61. Of the many instruments to which the principle of the lever is applied, balances or contrivances for weighing are perhaps those in most common use.

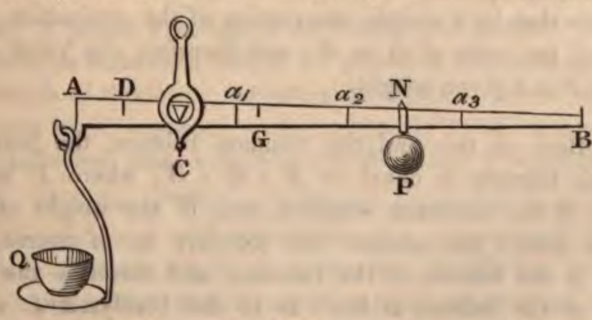
In all of them the adjustment is such that the arm at which the weight of the substance to be weighed acts, is either of constant or of known length; hence the weight itself will be known when its moment about the fulcrum is known; but this, by the principle of the lever, is equal to the moment of the counteracting force. Balances differ from one another in the methods which are adopted for making this latter moment known by inspection.

62. The most usual form of the balance is that termed the *Common Balance.* It is simply a heavy lever of the first kind,

whose arms are equal; the substance to be weighed, and the weight which is to counteract it, are placed in two scales, which are suspended, one from each extremity of the lever. When the balance is in proper adjustment, the weight of the arms and scales of themselves keep the lever or *beam* in equilibrium in a *horizontal* position, the centre of gravity of the whole being *below* the fulcrum: it is evident, therefore, that as the arms are equal, the substance weighed will have the same weight as the standard weight in the other scale, when the two keep the beam *horizontal:* and if the beam, thus in equilibrium, be disturbed, it will after a few oscillations return to its position; the quicker the beam assumes its horizontal position after such disturbance, the more *stable* is the balance termed. If the weights in the two scales be not equal, the beam will in its position of equilibrium be inclined to the horizon: when this angle of inclination is great for a small difference of the weights, the balance is said to possess great *sensibility.* The best form of the balance for insuring these requisites of *horizontality*, *stability*, and *sensibility*, can be discovered without much difficulty of investigation: the principles of parallel forces or of the lever, already discussed, are sufficient guides to the solution.

63. The next important form of the balance is the *Common Steelyard.*

It is represented in the annexed figure. AB is a graduated



lever suspended from a fixed fulcrum C. At A is a scale-pan

in which the substance to be weighed is placed, and P is a given weight which is moveable along AB, and its position on the graduated beam (when the beam is horizontal) determines the weight of any substance in the scale-pan.

Let W be the weight of the beam and scale-pan together, G their centre of gravity, and let Q be the weight of the substance to be weighed, and N the position of P when there is equilibrium.

Then since the resultant of P at N, W at G, and Q at A, must pass through C, the sum of their moments about C must vanish, and therefore

$$Q.AC = P.CN + W.CG.$$

If D be the point from which, if P be suspended, it will keep the beam in equilibrium, when not loaded with any weight in the scale-pan, then

$$P.CD = W.CG,$$

and therefore, by substitution in the above equation,

$$Q.AC = P.(CN + CD) = P.DN,$$

therefore $$Q = P\frac{DN}{AC}.$$

Suppose now that the beam DB be graduated by taking Da_1, Da_2, &c. equal to AC, $2AC$, &c., respectively; and suppose the figures 1, 2, 3, &c. to be placed over these points of graduation $a_1, a_2. \ldots,$ subdivisions being made between them: it is clear that by a simple observation of the graduation at N we know the ratio of Q to P; and therefore the value of Q itself as P is a given weight.

64. Both in this and the common balance, the pressure upon the fulcrum is equal to $P + Q + W$, where P is the weight, Q the substance weighed, and W the weight of the machine itself; the greater this pressure is, of course, the greater is the friction at the fulcrum, and therefore the sensibility of the balance is less: as in the Steelyard P never changes, while in the common Balance it must always equal Q

it is clear that the first has the advantage over the latter when used for determining weights greater than P; but that for smaller weights the common balance is better.

For the description of the Bent Lever Balance, Roberval's Balance, Danish Balance, and various other machines, reference may be made to the *Second Treatise on Mechanics*, published by the Society for the Diffusion of Useful Knowledge.

Examples to Section VI.

(1). A uniform rod two feet long, weighing 2 lbs., can turn about a fulcrum four inches distant from one extremity; and from the end nearest to the fulcrum a weight of 16 lbs. is suspended: find where a weight of 6 lbs. must be suspended in order to produce equilibrium.

Assume the point to be at a distance x feet from the fulcrum: then the resultant of the 6 lbs. acting here, the 16 lbs. at its extremity of the rod, and the weight of the rod itself at its middle point, must be counteracted by the resistance of the fulcrum; it must therefore pass through the fulcrum: and hence the sum of the moments of the three above forces taken about the fulcrum must vanish. This consideration gives an equation for finding x.

(2). A power P acting at one end of a lever three feet from the fulcrum balances a weight $3P$ placed at the other end: find the length of the lever, neglecting its weight. If the weight of the lever itself, supposed uniform, were $\frac{1}{2}P$, and its length the same as before, what weight would the power P support?

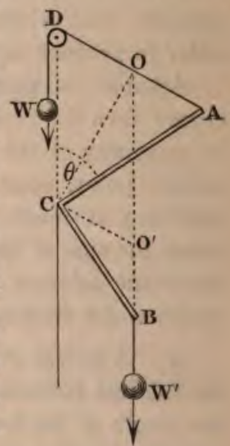
(3). Weights of 1 lb. and 4 lbs. are suspended from the ends of a straight lever without weight; the fulcrum and a point, at which another weight is suspended, divide the lever into three equal parts: find the third weight when there is equilibrium.

(4). A uniform heavy rod four feet long with a weight of 12 lbs. attached to one extremity balances upon a fulcrum nine inches from that extremity: find the weight of the rod.

(5). A weight P, when suspended successively at each end of an uniform rod, which is capable of turning freely about a fixed point in itself, is balanced by weights Q, R respectively at the other end: find the position of the fixed point, and the weight of the rod.

(6). The two arms of a weightless lever are at right angles to each other; to the one arm a weight is attached by means of a string passing over a pulley which is vertically over the fulcrum and at a distance from it equal to the length of the arm; to the other arm another weight is hung. Determine the position of equilibrium.

Let CA, CB be the two arms of the lever, D the pulley vertically above the fulcrum C, and such that $CD = CA$.



Let W be the weight attached to A by means of the string AD passing over the pulley, W' the weight suspended at B; then the bent lever is in equilibrium under the action of these two forces, and the resistance of the fulcrum at C; therefore the resultant of W and W' must pass through C; let the direction of W' meet that of W in O, then CO is the direction of their resultant, and the triangle CDO has its sides parallel to and therefore proportional to the three forces W, W', and their resultant therefore
$$W' : W :: CD : OD.$$

If DCA be represented by θ, we have, drawing CO' parallel to AD,
$$OD = CO' = BC\frac{\sin CBO'}{\sin CO'B},$$
$$= BC\frac{\cos\theta}{\cos\frac{\theta}{2}};$$

therefore, by substitution above, since $CD = CA$,

$$W' : W :: CA : CB \frac{\cos\theta}{\cos\frac{\theta}{2}};$$

therefore putting $CA = a$, $CB = b$, we get

$$bW' \cos\theta = aW \cos\frac{\theta}{2}.$$

The solution of this equation gives us

$$\cos\frac{\theta}{2} = \frac{\sqrt{(8b^2W'^2 + a^2W^2)} + aW}{4bW'} \dots\dots\dots (1),$$

or

$$= -\frac{\sqrt{(8b^2W'^2 + a^2W^2)} - aW}{4bW'} \dots\dots\dots (2):$$

of these values the negative one is always less than 1; the positive one is so when bW' is not less than aW. If then we consider the position of equilibrium as given by the result (1), we observe,

1*st*, that when bW' is less than aW, no such position is possible with the assumed conditions.

2*nd*, that $\theta = 0$, or CA coincident with CD, and therefore CB horizontal gives the position of equilibrium when $bW' = aW$.

3*rd*, that as the ratio of bW' to aW increases, θ also increases, or the angle which gives the position of equilibrium of CA also increases.

4*th*, that the greatest value, which this angle could have, would be, when the ratio of bW' to aW approached infinity; but in that case by (1) $\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$, therefore $\theta = \frac{\pi}{2}$, and CA would be horizontal.

If now we consider the equilibrium as given by the negative result (2), we observe,

(1), gives positions of equilibrium for all values of W and W'.

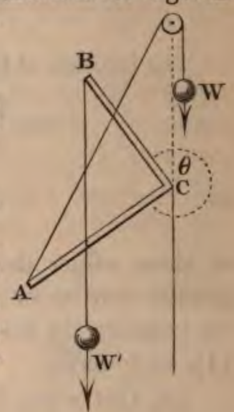
(2), that when W' equal nothing, the position of equilibrium will be when CA lies vertically downwards.

(3), that as W' increases, θ increases.

(4), that when bW' is very large compared with aW, θ becomes very nearly equal to $\frac{3\pi}{2}$, because $\cos\frac{\theta}{2}$ then approaches to $-\frac{1}{\sqrt{2}}$ as its value.

The first figure gives the position of equilibrium for a given value of W and W' as found from the result (1).

The annexed figure gives the position for the same value of W and W' as found from (2).

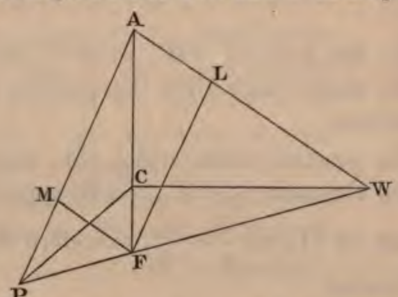


(7). If two forces P and W sustain each other on the arms of a bent lever PCW, whose fulcrum is C, and act in directions PA, WA, which form the sides of an isosceles triangle PAW; shew that if AC be joined, and produced to meet PW in F,

$$P : W :: FW : FP.$$

The annexed figure represents the system.

Draw FL, FM parallel to PA and WA respectively.



Since the lever is kept in equilibrium by the forces P, W, and the reaction at C, the resultant of P and W must pass through C; hence AC is its direction, and by the parallelogram of forces, P and W must be proportional to AM and AL respectively, or

$$P : W :: FL : FM.$$

Now, by construction it is evident that PMF and FLW are both isosceles triangles similar to each other, therefore

$$FL : FM :: FW : FP,$$

therefore $$P : W :: FW : FP.$$

(8). Two equal scale-pans are suspended from the ends of a straight lever, whose arms are as 3 and 4; and an iron bar of 20 lbs. weight is laid upon the scale-pans (just reaching from one to the other): in which of the scale-pans must a weight be placed, and how great must it be, in order to produce equilibrium?

The forces acting here are the weight of the beam and scale-pans, and the weight of the iron bar, both acting in a vertical line passing through the *middle* point of the lever, the weight required to be placed in the scale-pan nearest the fulcrum, and the resistance of the fulcrum.

(9). If the arm of a cork compresser be 18 inches long, and the cork be placed at an inch-and-a-half from the fulcrum, find the pressure produced by a weight of 12 stone suspended from the handle.

(10). The whole length of each oar of a boat is 10 feet, and from the hand to the rowlock the distance is 2 feet 6 inches; each of 8 men sitting in the boat pulls his oar with a force of 50 lbs. Supposing the blades of the oars not to move through the water, find the resultant force propelling the boat.

Each man pushes *back* the boat with his feet with a force exactly equal to that which he applies to the handle of the oar, *i.e.* 50 lbs.; but he presses it *forward* also by the force which his oar produces upon the rowlock; it is therefore the difference between these forces by which he really pushes the boat forward.

(11). What is the ratio between the radii of a wheel and its axle, when a cwt. balances a ton?

(12). Two weights P and W are supported on a wheel and axle, P by a string passing round the wheel, W by a moveable

pulley whose strings are parallel, and one of them winds on the wheel and off the axle, as P descends: determine the ratio of P to W.

This may be solved in two ways: the weight may be considered as applied at any point of the wheel which is vertically above it, in which case the tension of the string must be omitted, for it serves only to keep the weight rigidly attached to the wheel; or again, we may find the tension by considering the weight as supported on a single moveable pulley, and then consider P as kept in equilibrium by the application of this tension both at the wheel and at the axle.

(13). In a combination of wheels and axles, in which the circumference of each axle is applied to the circumference of the next wheel, and in which the ratios of the radii of the wheels and axles are $2:1$, $4:1$, $8:1$, and there is equilibrium when the power is to the weight as $1:p$; determine the number of wheels.

(14). If a man stand in a scale attached to a moveable pulley, and a rope having one end fixed pass under this pulley and then over a fixed pulley, with what force must he hold down the free end in order to support himself, the strings being parallel?

(15). P and W represent the power and weight upon the inclined plane: if P's direction lie between the vertical and the normal to the plane, shew that the body must be supported beneath the plane.

Assume R to be in the usual direction; its value will be found to be of a negative sign: the interpretation of this is not difficult.

(16). If the weight W could be supported on a single moveable pulley by a force P, what must be the inclination of a plane on which the same weight could be supported by the same force applied parallel to the plane?

(17). A smooth wedge of given vertical angle is inserted between a horizontal plane and a vertical screw: if a force W

be applied to the screw at an arm a, what must be the force acting upon the head of the wedge to preserve equilibrium?

(18). A smooth solid body in the shape of a wedge is placed with one side upon a horizontal plane, the other side thus forms an inclined plane; a heavy body is placed upon this inclined plane, and is prevented from sliding down it by a string attached to it, which is fixed to the top of the plain: why does not the pressure of the heavy body perpendicular to the side of the wedge make it slide along the horizontal plane?

(19). The arms of a balance are unequal and one of the scales is loaded; a body whose true weight is P lbs. appears to weigh W lbs. when placed in one scale, and W' lbs. when in the other: find the ratio between the arms, and the weight with which the scale is loaded.

(20). A shopkeeper has a false balance, and thinks to make his customers' consequent losses and gains balance each other by weighing the goods which he sells alternately in the one scale and in the other: does he succeed?

(21). The sliding weight of a Steelyard is 9 lbs. The zero point of graduation is $\frac{1}{2}$ an inch from the fulcrum on the longer arm, and the whole beam will balance about a point 3 inches from the fulcrum on the shorter arm: what is the weight of the beam?

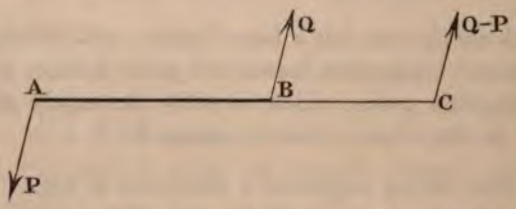
(22). In a common Steelyard whose moveable weight is one pound, it is observed that the distance between the point of suspension C and the graduation marked 1 lb. is exactly half the distance between C and the point A from which the substance to be weighed is hung; the steelyard alone weighs 6 lbs. Determine the distance of its center of gravity from C.

CHAPTER II.

SECTION I.

ON COUPLES AND SOME PROPOSITIONS RELATING TO FORCES NOT IN ONE PLANE.

65. We have seen, (Art. 23), that if P and Q, two parallel forces, of which Q is the greater, act in opposite direction at



points A and B rigidly connected together, their resultant will be a force $Q - P$ parallel to each of them, and acting at a point C in AB produced such that

$$AC = \frac{Q}{Q - P} AB.$$

If then P do not differ much from Q, $Q - P$ is very small compared with Q, and therefore AC is very large compared with AB; that is, the resultant of the two forces is a very small force, and its point of application is at a great distance from A or B.

If the difference between P and Q be still less, their resultant is again much smaller, and the distance of its point of application is greater.

By pursuing this train of investigation we are led to see that when P differs by an indefinitely small quantity from Q, their resultant is an indefinitely small force acting at an indefinitely great distance from A or B: or, in other words, when $P = Q$ the proposed system of forces can have no single resultant force, and cannot therefore be kept in equilibrium by the action of any single force.

DEF. Such a pair of forces is called a *Couple*. The perpendicular distance between the two forces of a couple is called the *arm* of the couple; and the product of the arm into the force is called the *moment* of the couple. It is also conventional to call the couple or its moment positive or negative, according as the forces appear to tend to turn the arm from left to right, or from right to left. This convention, it will be seen, always leads to consistent results.

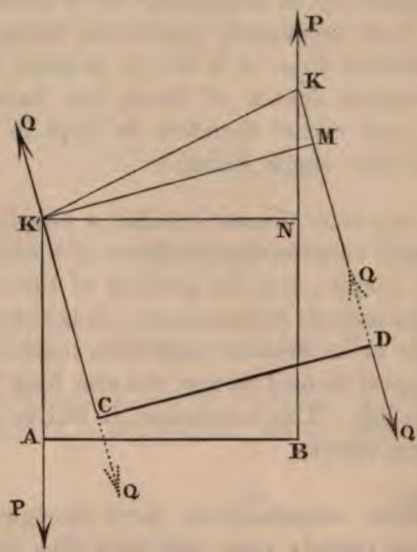
66. The above considerations have brought before us systems of forces entirely new, and have thus imposed upon us the task of investigating the laws which govern their combination: no new physical assumptions are however here necessary: the proposition of the parallelogram of forces and the principle of the transmission of force through a rigid body are sufficient data for our purpose.

67. By the aid of the *Definition* "That one couple is equal or equivalent to another when its forces, being reversed in direction, and their points of application being rigidly connected with the points of application of the forces of the other, will keep those of the other in equilibrium," we may assert the fundamental proposition as regards couples in one plane in the following terms:

A couple placed anywhere in the same plane with a given couple, will be equal to it when the moments of the two are equal.

The two couples may either have their arms parallel or inclined to each other.

First. Let them be inclined to each other, as in the



annexed figure, where AB is the arm of the first couple, P the force, CD the arm of the second couple, Q its force.

The dotted arrow-heads represent the direction of the forces of the second couple before reversion.

It is evident that the forces of the reversed couple must meet those of the first; let them do so in k and k'. If k and k' be supposed rigidly connected with the system of points AB, CD, we may conceive them to be the points of application of the two pairs of forces P and Q, (Art. 15). If the couple CD, Q, be equal to AB, P, then from definition the forces of the reversed couple must balance those of the first, that is, the P and Q acting at k must have a resultant equal and opposite to that of P and Q at k'.

Now the angle PKQ is equal to the angle $PK'Q$, because KQ is parallel to $K'Q$ and KP to $K'P$; hence the resultant of P and Q at K must be equal and parallel to the resultant of P and Q at K': it is only necessary therefore that KK' should be its line of direction.

This will be the case if

$$\frac{P}{Q} = \frac{\sin K'KD}{\sin K'KP}$$

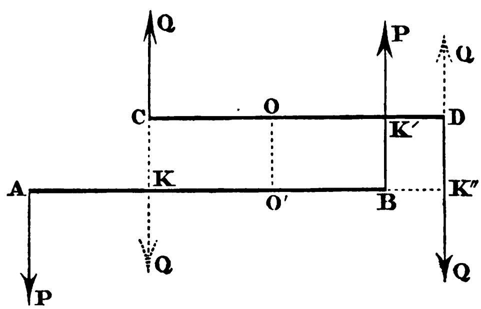
$$= \frac{KK' \sin K'KM}{KK' \sin K'KB}$$

$$= \frac{CD}{AB},$$

i.e. $$P.AB = Q.CD,$$

or if the moments of the two are equal.

Secondly. Let the arms be parallel, as in the annexed figure, where as before the dotted lines represent the direction



of the forces of the second couple before reversion; let one of these meet AB or AB produced in K; also let the forces P at B and Q at D meet CD and AB respectively, or these produced in K' and K''.

We may conceive the points C, K', A, K'' to form one rigid system to which the above forces are applied; and which must be kept in equilibrium by them, from definition, when the couples are equal.

Now the resultant of Q at C and P at K' is a force $P + Q$ parallel to each, acting at a point O in CD between them such that $DO = \dfrac{P.DK' + Q.DC}{P + Q}$. (Art. 24).

Again, the resultant of P at A and Q at K'' is a force

$P+Q$ parallel to each, acting at O' at point in $K''A$, such that

$$K''O' = \frac{P}{P+Q}\,K''A$$
$$= \frac{P.K''B + P.AB}{P+Q}$$
$$= \frac{P.DK' + P.AB}{P+Q}.$$

Hence it is evident that the only thing necessary for equilibrium is, that

$$K''O' = DO,$$

or that
$$P.AB = Q.DC,$$

or, as before, that the moments of the couples be equal.

68. If any number of couples are acting upon a system in the same plane, they may be replaced by one couple whose moment is equal to the sum of their moments.

Let the moments of the given couples be P_1a_1, P_2a_2, ... P_na_n, *i.e.* let their arms be respectively $a_1a_2...a_n$ and their forces $P_1P_2...P_n$.

For each of these we may, by the last Article, substitute a new couple with an arm equal b, provided we also alter the forces so that the moments of each of the new ones shall be equal to that of the one which it replaces, *i.e.*

the force of the first new one must equal $\frac{a_1}{b}P_1$,

.... second .... $\frac{a_2}{b}P_2$,

and of the nth new one $= \frac{a_n}{b}P_n$.

Again, as by the last Article, it is a matter of indifference what is the position of a couple in its own plane: let all these new ones be so placed that their arms each equal to b coincide. When thus arranged it is evident that there will

be a force at each end of this arm b which is made up of all the above forces; it is also clear that the forces belonging to the negative couples will be opposite to the forces of the positive couples; therefore, calling the resultant force at each end of the arm b, Q, we have

$$Q = \frac{a_1}{b} P_1 + \frac{a_2}{b} P_2 + \ldots\ldots + \frac{a_n}{b} P_n,$$

where the second side is the *algebraical* sum of the above-mentioned forces, considering those positive which belonged to the originally positive couples, and those negative which belonged to the negative couples. Hence

$$bQ = a_1P_1 + a_2P_2 + \ldots\ldots + a_nP_n$$

equals the algebraical sum of the moments of the original couples: but the force Q acting at each end of the arm b forms a new couple whose moment is bQ. Therefore if there be any number of couples in one plane,

The couple whose moment is equal to the algebraical sum of the moments of the given couples is in every way equivalent to them.

69. From what has just been proved, the couple which is equivalent to two other equal couples applied together, has its moment equal to the sum of their moments. In other words, the moment of a couple which is double of another is double the moment of that other.

Similarly, a couple which is equivalent to three equal couples, *i.e.* which is treble of any one of them, has a moment which is three times the moment of that one.

By proceeding in the same way it could be shewn that a couple which is n times another couple has a moment equal to n times the moment of this one. Hence we see that couples are proportional to their moments, and may therefore be properly measured by them: thus, if C represent a couple whose moment is M,

$$C = aM,$$

where a is some constant quantity for all couples. If a couple whose moment equal 1, be taken to equal 1, or be taken for the unit of couple in terms of which all others shall be measured, we must have from above

$$1 = a,$$

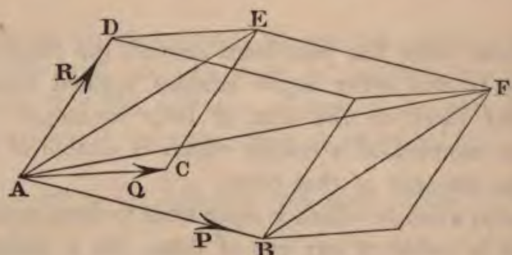
therefore on this supposition

$$C = M,$$

or a couple is always represented by its moment. The unit of couple spoken of may very well be the couple whose force is equal to the unit of force and whose arm is equal to the unit of length, for such a couple would necessarily have its moment equal 1.

70. If three forces act upon a point in space, and lines be drawn from that point parallel and proportional to them, their resultant is parallel and proportional to the diagonal of the parallelopiped described upon these three lines as edges.

Let the three forces P, Q, R act upon the point A, and



let the lines AB, AC, AD be taken parallel and proportional to them respectively. Upon these three lines as edges complete the parallelopiped, $ACEFB$; then the resultant of P, Q and R acts along, and is in the same proportion to AF, as P, Q, and R are to AB, AC, AD respectively.

Join AC, BF.

Because AE is the diagonal of the parallelogram described upon AC, AD, it is in the direction of and proportional to the resultant of Q and R.

Again, because AF is the diagonal of the parallelogram described upon AE, AB, it is parallel and proportional to the resultant of the forces represented by them, *i.e.* it represents in magnitude and direction the resultant of the force P and the resultant of Q and R: it therefore represents the resultant of the three given forces.

71. The results of Articles (32) and (37) may be included in the following general enunciation.

If a body be in contact with a smooth plane at any number of points, and these points be joined successively so as to form a polygonal figure, since the resistances of the plane upon the body at these points form a system of parallel forces acting in the same direction, their resultant will evidently be parallel to them, and will pass through some point within the polygonal figure. Hence for equilibrium the resultant of the other forces acting upon the body must be perpendicular to and act towards the plane, and must pass through some point within the above-mentioned polygonal figure. It is indifferent whereabouts within the figure this point be situated, for the resistances at the points of contact are indeterminate and may be exerted to any required amount; and in all cases just so much force will be called into action at each point as will make the resultant resistance act at the same point as the resultant of the other forces; only this is not possible when the latter does not fulfil the condition of passing within the polygonal figure.

If the resultant of the other forces be perpendicular to the plane, but fall without the figure of contact, the body will turn over. If it be not perpendicular to the plane, but yet pass through the figure, the body will slide. If it be neither perpendicular nor yet pass through the figure, the body will begin to both slide and turn over.

72. When gravity is the only force acting upon the body besides the resistances of the plane, the foregoing proposition reduces itself to the assertion that the vertical line through the centre of gravity of the body must be perpendicular to the plane, and must not fall withoutside the polygonal figure formed by joining the successive points of contact.

SECTION II.

VIRTUAL VELOCITIES.

73. A large class of Statical Problems may be easily solved by the aid of an artifice which is termed the Principle of Virtual Velocities. Although it does not strictly belong to geometrical Statics, it deserves to be mentioned on account of some of the remarkable results to which it leads us.

DEF. Suppose any number of forces to be acting at different points of a body, and suppose the body to receive an *indefinitely* small displacement: if now perpendiculars be drawn from the new positions of the points of application of the forces upon the directions of the forces as they were before the displacement, the line intercepted between the foot of any one perpendicular and the first position of the point of application of the corresponding force is called the virtual velocity of that force, and is estimated as positive or negative according as it falls on the side of the point towards which the force acts, or the contrary.

74. The Principle of Virtual Velocities asserts that if any number of external forces acting upon a body or system of points, be in equilibrium, then the algebraical sum of the terms formed by multiplying each force by its virtual velocity vanishes: thus if $P_1 P_2 \dots P_n$ represent the forces, $p_1 p_2 \dots p_n$ their respective virtual velocities when the body has received any given indefinitely small displacement whatever, if $P_1 P_2 \dots P_n$ are in equilibrium with each they must satisfy other the relation

$$P_1 p_1 + P_2 p_2 + \dots + P_n p_n = 0 \dots\dots\dots (1).$$

It is difficult, or perhaps impossible, to give a general proof of this assertion, based solely upon the properties of force and independent of its particular mode of action; but its truth has

been ascertained for almost every conceivable arrangement or method by which forces may act upon a system of points: a few only of the simplest cases will be given here in which it is verified, and a few examples will be solved by its application.

75. If the supposed displacement be made in such a manner that some of the virtual velocities equal zero, *i.e.* that some of the p's in equation (1) vanish, the same number of terms, with their corresponding forces, will disappear from the equation: the analytical relation which the remaining forces must satisfy will therefore be simplified. In using equation (1) it would always be our object to make the displacement so as to get rid of the forces whose value we do not care to find: theoretically there is no difficulty in doing this, but in practice all displacements do not afford equal facilities for finding the geometrical quantities $p_1 p_2 \ldots\ldots p_n$; we are obliged to choose those which are most convenient. It is therefore readily seen that the Principle of Virtual Velocities is most useful in those cases where the simplest displacement at the same time retains only those forces which we desire chiefly to consider: such happens when some of the forces are supplied by contact with smooth surfaces; for if the displacement be made by sliding the system along them, the virtual velocities of the reactions are evidently nothing, and the forces themselves disappear from the above relation.

76. If $P_1 P_2 \ldots P_n$ be any number of forces acting in the same plane at the same point, R their resultant, $p_1 p_2 \ldots p_n$, r their respective virtual velocities consequent upon a displacement of the point made in that plane; then, generally,

$$P_1 p_1 + P_2 p_2 + \ldots\ldots + P_n p_n = Rr.$$

First let us consider only the first two forces as $P_1 P_2$ acting upon the point A, and let AB, AC represent them in magnitude and direction; their resultant R_1 will be represented by AD the diagonal of the parallelogram described upon AB, AC. Suppose A to be displaced to a position A', and join AA'.

AA' may make acute angles with both AB and AC, or an acute angle with one and an obtuse with the other, or it may make obtuse angles with both. In the first of the annexed figures it makes acute angles with both, in the second it makes acute angles with AC and AD, but an obtuse angle with AC.

Fig. 1.

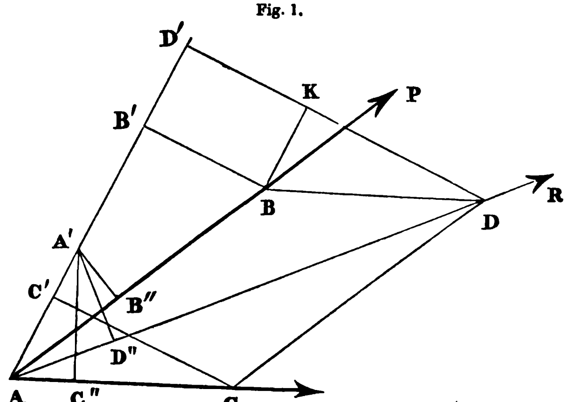
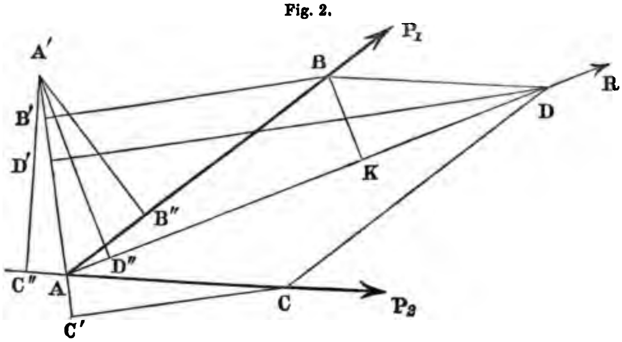


Fig. 2.



Draw BB', CC', DD' perpendicular to AA' or AA' produced if necessary, and also $A'B''$, $A'C''$, $A'D''$ perpendiculars to AB, AC, AD, respectively, or if necessary to those produced. Draw BK parallel to AA' meeting DD' in K.

Then, from similar and equal triangles ACC', BDK, $BK = AC' = B'D'$. Hence, in the first figure,

$$AB' + AC' = AD' \ldots\ldots\ldots\ldots\ldots (1),$$

in the second,* $$AB' - AC' = AD' \ldots\ldots\ldots\ldots\ldots (2).$$

* BK is drawn incorrectly.

Again, because the triangles $AA'B''$, $AA'C''$, $AA'D''$ are respectively similar to triangles ABB', ACC', ADD', therefore

$$AB' = AB\,\frac{AB''}{AA'},$$

$$AC' = AC\,\frac{AC''}{AA'},$$

$$AD' = AD\,\frac{AD''}{AA'}.$$

Hence, by substitution in (1) and (2), we have

$$AB.AB'' + AC.AC'' = AD.AD'', \text{ for first figure.}$$

$$AB.AB'' - AC.AC'' = AD.AD'', \text{ for second figure.}$$

Now AB, AC and AD, are respectively proportional to P_1, P_2, and R, and AB'', AC'', AD'' are, by definition of virtual velocities, proportional to p_1, p_2, and r_1, for they would themselves be the virtual velocities if AA' were indefinitely small, the p_2 in the second figure being negative from the same definition; therefore the above equations both become

$$P_1p_1 + P_2p_2 = R_1r_1. \ldots\ldots\ldots\ldots\ldots (3).$$

If AA' had made an obtuse angle with the directions of two or all of the three forces P_1, P_2, R, it is clear by the geometry of the given figures, that the corresponding displacement would have been negative, as is the second term of equation (2); but the algebraical sign being included in the symbol itself of the virtual velocity, when the substitution is made all cases are included in the form (3).

Suppose now R_2 to be the resultant of R_1 and P_3, r_2 being its virtual velocity; then, by the form (3),

$$R_1r_1 + P_3p_3 = R_2r_2,$$

or

$$P_1p_1 + P_2p_2 + P_3p_3 = R_2r_2.$$

By proceeding in this manner we shall arrive at last at the result proposed for proof, that

$$P_1p_1 + P_2p_2 + \ldots\ldots + P_np_n = Rr \ldots\ldots\ldots (4).$$

It is to be observed that this equation (4) is in this case equally

true whether the quantities $p_1 p_2 \ldots\ldots$ represent indefinitely small or finite displacements.

77. When the forces $P_1 P_2 \ldots\ldots P_n$ are in equilibrium R must vanish, and therefore equation (4), in this case, becomes

$$P_1 p_1 + P_2 p_2 + \ldots\ldots + P_n p_n = 0,$$

which proves the Principle of Virtual Velocities to be true for forces acting in one plane upon the same point.

78. To prove the equation when the forces $P_1, P_2 \ldots P_n$ all act in the same plane but not at the same point of a rigid system.

Suppose $A_1, A_2 \ldots A_m$ to be the points of application of these forces, and let

$$\begin{aligned} &P_1, P_2 \ldots\ldots P_{s_1} \text{ act at } A_1, \\ &P_{s_1+1} \ldots\ldots P_{s_2} \ldots\ldots A_2, \\ &P_{s_{m-1}+1} \ldots\ldots P_n \ldots\ldots A_m. \end{aligned}$$

We may conceive the rigidity of the system to be preserved by any kind of mutual reactions of the points $A_1, A_2 \ldots A_m$ upon each other, for instance they may be connected by rigid rods without weight, jointed freely at the particles; in this case the reactions would be in the lines joining the particles pair and pair together: and it would seem therefore generally sufficient to consider them so to act. Let the reaction of any one of the points, as A_r, upon any other, as A_q, be represented by ${}_rT_q$, and its virtual velocity by ${}_rt_q$.

Then, since the point A_1 is kept in equilibrium by the forces $P_1, P_2 \ldots P_{s_1}, {}_2T_1, {}_3T_1 \ldots {}_mT_1$, whose virtual velocities are respectively $p_1, p_2 \ldots p_{s_1}, {}_2t_1, {}_3t_1 \ldots {}_mt_1$, we get from equation of Art. 77,

$$P_1 p_1 + P_2 p_2 + \ldots + P_{s_1} p_{s_1} + {}_2T_1\, {}_2t_1 + {}_3T_1\, {}_3t_1 + \ldots + {}_mT_1\, {}_mt_1 = 0 \ldots (1).$$

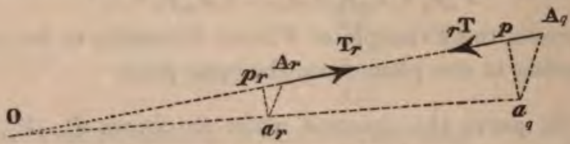
Similarly, for equation of A_2,

$$P_{s_1+1} p_{s_1+1} + \ldots + P_{s_2} p_{s_2} + {}_1T_2\, {}_1t_2 + {}_3T_2\, {}_3t_2 + \ldots + {}_mT_2\, {}_mt_2 = 0 \ldots (2),$$

and so on for each of the other points. And lastly for A_m,

$$P_{s_{m-1}+1} p_{s_{m-1}+1} + \ldots + P_n p_n + {}_1T_m\, {}_1t_m + {}_2T_m\, {}_2t_m + \ldots + {}_{m-1}T_m\, {}_{m-1}t_m = 0 \ldots (m).$$

Now, since the reaction of A_r upon A_q is equal to that of A_q upon A_r, we must always have ${}_rT_q = {}_qT_r$. Also it can be proved that generally the virtual velocity of ${}_rT_q$ is equal but opposite to that of ${}_qT_r$, therefore ${}_rt_q = -{}_qt_r$: for let the annexed figure represent the relative positions of A_r and A_q; a_r, a_q their posi-



tions after an indefinitely small displacement; a_rp_r, a_qp_q perpendicular to the line joining A_rA_q. If A_ra_r be parallel to A_qa_q, A_rp_r must equal A_qp_q, for a_ra_q is equal to A_rA_q: and $A_qp_q = {}_rt_q$, $A_rp_r = -{}_qt_r$, therefore ${}_rt_q = -{}_qt_r$.

If A_rA_q be not parallel to A_ra_q, let them meet in some point O; then, since the displacements are indefinitely small, p_ra_r, p_qa_q are very approximately the arcs of circles described about O as centre with radii Oa_r and Oa_q respectively, therefore

$$\begin{aligned} A_rp_r &= OA_r - Op_r \\ &= OA_r - Oa_r, \end{aligned}$$

and
$$\begin{aligned} A_qp_q &= OA_q - Oa_q \\ &= (OA_r + A_rA_q) - (Oa_r + a_ra_q), \\ &= OA_r - Oa_r, \quad \because A_rA_q = a_ra_q, \\ &= A_rp_r; \end{aligned}$$

therefore, as before, ${}_rt_q = -{}_qt_r$.

Hence we arrive at the general result, that

$${}_rT_q\,{}_rt_q + {}_qT_r\,{}_qt_r = 0.$$

If now we add together the equations (1), (2). . . .(m), we find that this makes all the terms involving the mutual reactions to disappear, and we obtain

$$P_1p_1 + P_2p_2 + \ldots + P_np_n = 0 \ldots\ldots\ldots (A),$$

which asserts the principle of virtual velocities for forces acting at different points of a rigid system in the same plane.

79. It is worth observing, that since equations (1), (2)...(m) always hold, equation (A) will also always be true as long as the terms involving the reactions disappear in the addition: this will be the case equally when the points A_1, $A_2 \ldots . A_n$ are rigidly connected, as supposed in the preceding proof, and when they are not connected at all, for then certain pairs of T's corresponding to pairs of unconnected points will each be nothing. On the contrary, equation (A) will not be true whenever the virtual velocity of the mutual force between any two given points is not of the same magnitude and of different sign for both those points; for instance, A_r and A_q in the above illustration might be two points of an elastic body which in the displacement are made to approach each other in such a manner that $A_r p_r$ is of same sign as $A_q p_q$, and therefore ${}_rT_q\,{}_rt_q + {}_qT_r\,{}_qt_r$ does not vanish; and consequently the equation that would result from the addition of (1), (2). . . .(m) would contain some if not all of the unknown internal tensions.

It appears then that in order that the equation of virtual velocities may hold, the displacement of the system must be made in such a way that the *geometrical* connexions existing between the points at which the external forces act, shall remain undisturbed.

80. Conversely, if the equation of virtual velocities holds, *i.e.* if

$$P_1p_1 + P_2p_2 + \ldots . + P_np_n = 0 \ldots \ldots . . (1),$$

when $P_1 \ldots P_n$ are all the external forces acting upon a system of points, then must the system be in equilibrium. For if not, these forces must have either a single resultant, or else two resultants equal and opposite to each other and forming a couple.

In the first case, let R be the single resultant. If now it be applied to the system in an opposite direction it must necessarily keep it in equilibrium; hence if r be its virtual velocity when that of the P's is p,

$$P_1p_1 + P_2p_2 + \ldots . + P_np_n + Rr = 0 \ldots \ldots (2),$$

by the principle of virtual velocities.

By equation (1) this becomes $Rr = 0$, or since r may be anything indefinitely small, not necessarily zero, $R = 0$; there-

fore the given system of forces has no resultant, they are in equilibrium. In the second case, let Q, Q be the forces of the couple, to which the forces $P_1 ... P_n$ are equivalent, acting at the extremities of a certain arm; if forces equal to these be applied at the same points but in opposite directions, they will with $P_1 ... P_n$ keep equilibrium; hence, if q, q' be their virtual velocities when those of P are p, we have as before

$$P_1 p_1 + P_2 p_2 ... + P_n p_n + Qq + Qq' = 0 \ldots\ldots (3),$$

which by equation (1) becomes

$$Q(q + q') = 0:$$

this must be true for all displacements, but there is only one, that is, when the two extremities of Q's arm move parallel to each other, which will make $q + q' = 0$, therefore generally

$$Q = 0,$$

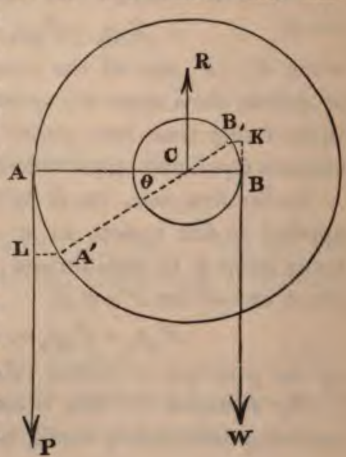
or there can be no resultant couple.

If therefore the equation of virtual velocities holds for forces acting in one plane, there must in all cases be equilibrium.

It is easy to verify the principle of virtual velocities in the case of the simple mechanical powers.

81. First, *it may be shewn to hold for the wheel and axle when the power and weight are in equilibrium.*

The figure represents a section of the wheel and axle perpendicular to its axis. The external forces acting upon the system are P and W acting vertically downwards at A and B respectively, and R the resistance of the point C, about which the machine turns.



Let now a small displacement be produced by turning the wheel and axle through a very small angle, θ about C, so that A and B come into the positions A' and B' respectively; draw $A'L$ and $B'K$ perpendicular to AP and BW

respectively, or, if necessary, to these produced. Then it is evident that the force R has received no virtual velocity, while those of P and W are AL and $-BK$ respectively.

Since θ is very small,

$$\left.\begin{array}{l} AL = AA' \\ BK = BB' \end{array}\right\} \text{very approximately;}$$

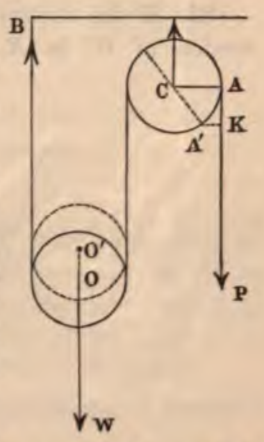
therefore
$$\begin{aligned} P.AL - W.BK &= P.AA' - WBB', \\ &= P.AC.\theta - W.BC.\theta, \\ &= (P.AC - W.BC)\theta, \\ &= 0 \ldots\ldots\ldots\ldots\ldots\ldots(1); \end{aligned}$$

because when there is equilibrium between P and Q on the wheel and axle,

$$P.AC = W.BC.$$

82. It holds for the single moveable pulley with the chords parallel when the power and weight are in equilibrium.

Let O be the centre of the moveable pulley, to which the weight W may be supposed to be attached, C the centre of the fixed pulley, A the point in its circumference where the string by which P is applied ceases to touch.



W applied at O, P at A, the strain at B the first extremity of the string, and the resistance of the pivot C are all the external forces which act upon the system, for the tensions at different points of the string, and the pressures between it and the pulley, are manifestly mutual and internal forces. Suppose now an indefinitely small displacement to be made, such that A revolves about C through a very small angle θ into the position A', and that O comes to O'; from A' draw $A'K$ perpendicular to AP; the forces at B and C have thus received no virtual velocity, while that of P is AK and that of W is $-OO'$.

Since θ is very small, $AK = AA'$ nearly,

$$= CA.\theta.$$

Also the string wound off the pulley O clearly is the same as that wound off $C = CA.\theta$, and as this produces a shortening in both strings of O, and OO' must be equal to the shortening of either, therefore $OO' = \frac{1}{2}CA.\theta$. Hence we find for the sum of the products of the virtual velocities into their forces,

$$P.AK - W.OO' = P.CA.\theta - W.\frac{CA}{2}.\theta,$$

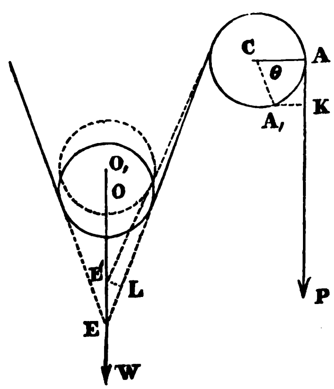
$$= \left(P - \frac{W}{2}\right) CA.\theta,$$

$$= 0;$$

because when there is equilibrium $P = \dfrac{W}{2}$.

In a similar way it may be proved to be true for any of the systems of pullies which have their strings parallel.

83. If the strings be not parallel, let them meet the direction of W in E, and after displacement let E' be the



position of E; draw $E'L$ perpendicular to EM, then $O'O = E'E$ approximately

$$= \frac{EL}{\cos OEL};$$

but $EL = \frac{1}{2}AK$ because it is half the shortening of the string; therefore

$$O'O = \frac{AK}{2\cos OEL},$$

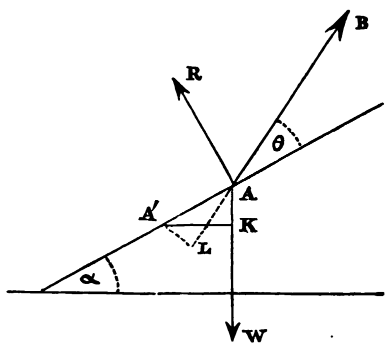
and $$P.AK - W.OO' = \left(P - \frac{W}{2\cos OEL}\right)AK,$$
$$= 0,$$

because when there is equilibrium

$$W = 2\cos OEL.P.$$

84. It holds for the *inclined Plane.*

Suppose the particle A to be kept in equilibrium upon the plane by a force P inclined at an angle θ to the plane. The



external forces acting upon A are P, W, and R the reaction of the plane; let the small displacement be made by sliding A along the plane into a neighbouring position A'; draw $A'K$, $A'L$ perpendicular to AW and AP; then R has received no virtual velocity, and that of P is $-AL$, that of W is AK.

Now $AK = AA' \sin\alpha$, where α is the inclination of the plane to the horizon, and

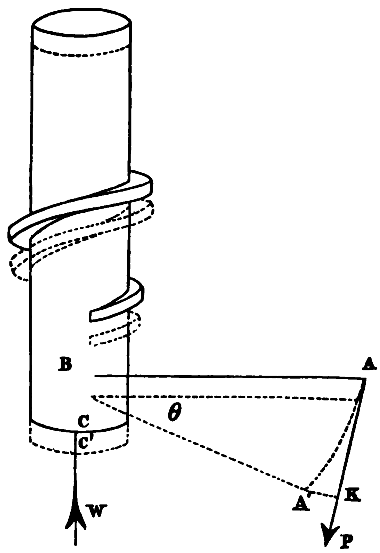
$$AL = AA'\cos\theta,$$

therefore $$W.AK - P.AL = (W\sin\alpha - P\cos\theta)AA',$$
$$= 0,$$

because when there is equilibrium, $W\sin\alpha = P\cos\theta$.

85. *It holds for the Screw.*

The external forces acting upon the screw are P acting perpendicular to the arm AB at extremity A, W acting in



the axis of the screw at C suppose, and the resistances at the different points of the thread normal to the thread.

Let an indefinitely small displacement be made by causing the screw to revolve through a very small angle θ, and let A come into the position A', and C to C'; draw $A'K$ perpendicular to AP. It is evident that the resistances at the different points of the thread have received no virtual velocities, while that of W is $-CC'$, and that of P is AK.

Now
$$AK = AA' \text{ nearly,}$$
$$= AB.\theta,$$

and $CC' = \dfrac{\theta}{2\pi} \times$ distance between two threads;

therefore

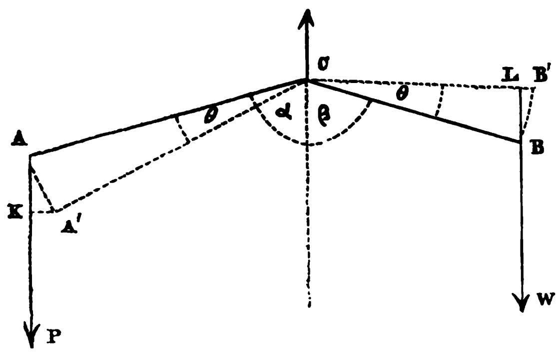
$$P.AK - W.CC' = \left\{P.AB - W\frac{\text{distance between two threads}}{2\pi}\right\}\theta,$$
$$= 0,$$

because when there is equilibrium,

$$P = W\frac{\text{distance between threads}}{2\pi AB}.$$

86. *It holds for a Lever of any form with forces parallel.*

Let AC, BC be the arms making angles α and β respectively with the directions of the power and weight.



The external forces acting upon the system are P, W at A and B, and the resistance of the fulcrum at C.

Let the displacement be made by turning the lever through a very small angle θ about C, so that A comes to A' and B to B'; draw $A'K$, $B'L$ perpendicular to AP and BW respectively. R has received no displacement, that of P is AK, and that of W is $-LB$.

Now $\quad AK = AA' \cos A'AK$ approximately,

$$= CA.\theta.\sin\alpha,$$

similarly $\quad BL = BC.\theta.\sin\beta;$

therefore $\quad P.AK - W.BL = (P.AC\sin\alpha - W.BC.\sin\beta)\theta,$

$$= 0,$$

because when there is equilibrium, $\dfrac{P}{W} = \dfrac{BC\sin\beta}{AC\sin\alpha}$.

The same may be proved when the power and weight are not parallel.

87. In all the preceding cases, the principle of virtual velocities has been proved to hold when equilibrium exists between the power and weight acting upon the machine; that is, if p and w represent the *actual* indefinitely small spaces unaffected by sign described by the power and weight respectively, in consequence of the indefinitely small displacement of the system, then

$$Pp = Ww \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In the Wheel and Axle, the Pullies with parallel strings, the Inclined Plane, the Screw, and the Lever with straight arms and parallel forces, the power and weight when once in equilibrium are in equilibrium for all positions of the machine, they will therefore be in equilibrium in the displaced position: if therefore we suppose the machine to receive a still farther indefinitely small displacement, w_1 and p_1 being the spaces described by W and P respectively, we must have as before

$$Pp_1 = Ww_1 \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and if we suppose this to be repeated n times, we get

$$Pp_2 = Ww_2 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\&c. = \&c.$$

$$Pp_{n-1} = Ww_{n-1} \ldots\ldots\ldots\ldots\ldots (n);$$

and hence by addition

$$P(p + p_1 + \ldots . + p_{n-1}) = W(w + w_1 + \ldots . + w_{n-1}).$$

Now these successive displacements may be repeated often enough to make P altogether describe a finite space S_p, while W describes a finite space S_w, *i.e.*

$$p + p_1 + \ldots . + p_{n-1} = S_p,$$

$$w + w_1 + \ldots . + w_{n-1} = S_w,$$

and therefore $$PS_p = WS_p.$$

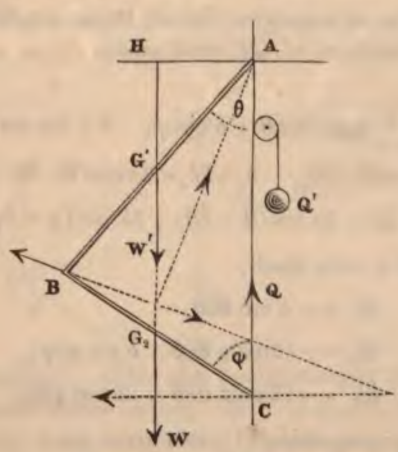
From this we learn that in these machines if a power P be employed to raise a weight W through any space, then *the product of the power into the space it describes, is equal to the*

product of the weight into the space it describes. Thus, if a man raise a bucket from a well by the aid of an ordinary wheel and axle, the force which he applies by his hand to the handle of the wheel multiplied by the space which his hand describes in turning the wheel, is equal to the weight of the bucket multiplied by the space through which it has ascended in the same time; the work will be made easier by increasing the size of the wheel, but the distance to be described by the hand will be thereby increased in the same proportion.

If the force be always supposed to describe the same space in the same time, it will be easily seen that "what is gained in power in any of the above machines is lost in time:" this is sometimes stated as a mechanical principle.

Examples to Section II.

(1). AB, BC are two heavy beams jointed together at B, and attached by a hinge at A to a vertical smooth wall, against



which BC rests at C. A string is also fastened at C, passes

over a pulley in the wall vertically above C, and carries a weight Q at its farther extremity. Find the position of equilibrium of the beams.

The whole system is in equilibrium under the action of the following forces:

The reaction of the wall, and Q, at C.

The weight W_2 of BC acting at its middle point G_2.

The reactions of the beams upon one another at B.

The weight W_1 of AB acting at its middle point G_1.

And the action of the hinge at A.

If we give an indefinitely small displacement to the whole system by sliding C along the wall, the only forces of these which will receive a virtual velocity will be W_1, W_2, and Q.

Let $BAC = \theta$, $BCA = \phi$, $AB = 2a$, $BC = 2b$, and let the distances of G_1, G_2, and C from a horizontal line through A be $HG_1 = h_1$, $HG_2 = h_2$, $AC = k$. When the system is displaced, let the altered values of θ, ϕ, h_1, h_2, k be $\theta + \delta\theta$, $\phi + \delta\phi$, $h_1 + \delta h_1$, $h_2 + \delta h_2$, $k + \delta k$ respectively, then δh_1, δh_2, δk are evidently the virtual velocities of W_1, W_2, and Q respectively; and we have by the principle of virtual velocities,

$$W_1\delta h_1 + W_2\delta h_2 - Q\delta k = 0 \;\ldots\ldots\ldots\ldots (1),$$

the last term being negative, for δk being considered positive must fall in direction of QC produced.

Now

$$h_1 = a\cos\theta, \quad h_2 = 2a\cos\theta + b\cos\phi, \quad k = 2a\cos\theta + 2b\cos\phi,$$

$$\therefore\; h_1 + \delta h_1 = a\cos(\theta + \delta\theta), \quad h_2 + \delta h_2 = 2a\cos(\theta + \delta\theta) + b\cos(\phi + \delta\phi),$$

$$k + \delta k = 2a\cos(\theta + \delta\theta) + 2b\cos(\phi + \delta\phi);$$

hence, since $\delta\theta$ is very small,

$$\delta h_1 = -\,a\sin\theta\delta\theta,$$

$$\delta h_2 = -\,(2a\sin\theta\delta\theta + b\sin\phi\delta\phi),$$

$$\delta k \;= -\,(2a\sin\theta\delta\theta + 2b\sin\phi\delta\phi).$$

Substituting in equation (1), this gives us

$$a(W_1 + 2W_2 - 2Q)\sin\theta\delta\theta + b(W_2 - 2Q)\sin\phi\delta\phi = 0 \ldots (2).$$

But by reference to the figure, we observe that

$$a\sin\theta = b\sin\phi \ldots\ldots\ldots\ldots\ldots(3),$$

and therefore $\quad a\sin(\theta+\delta\theta) = b\sin(\phi+\delta\phi),$

which gives us by subtraction

$$a\cos\theta\delta\theta = b\cos\phi\delta\phi:$$

hence substituting in (2),

$$(W_1 + 2W_2 - 2Q)\,b\cos\phi + (W_2 - 2Q)\,a\cos\theta = 0,$$

or $\quad (W_2 - 2Q)\,a\cos\theta = -(W_1 + 2W_2 - 2Q)\,b\cos\phi \ldots(4);$

therefore multiplying (3) by $(W_1 + 2W_2 - 2Q)$, and squaring (3) and (4), and adding, we have

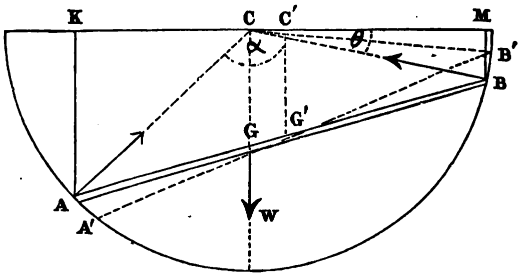
$$a^2\{(W_1+2W_2-2Q)^2\sin^2\theta + (W_2-2Q)^2\cos^2\theta\} = b^2(W_1+2W_2-2Q)^2,$$

therefore $\quad \tan^2\theta = \dfrac{b^2(W_1+2W_2-2Q)^2 - a^2(W_2-2Q)^2}{(a^2-b^2)(W_1+2W_2-2Q)^2}.$

This may be slightly simplified by taking W_1 and W_2 as proportional to a and b.

(2). A beam AB has its two extremities resting within a smooth circular hoop whose plane is vertical, find the position of equilibrium.

Let C be the centre of the hoop; α the angle which the beam subtends at C; θ the inclination of CB to the horizontal



diameter through C; BM, AK perpendicular to this diameter.

The external forces acting upon AB are the normal resistances of the hoop at A and B, and its own weight W at G.

Let a displacement be given to the system by sliding A and B along the curve into new positions $A'B'$, so that θ is diminished by the indefinitely small quantity $\delta\theta$, and G comes into the position G'.

It is clear that W only of the above forces has thus received a virtual velocity; if h be the original distance of G below KCM, and h' that of G', this virtual velocity $= h' - h$, and therefore, by virtual velocities, we must have

$$W(h' - h) = 0,$$

or
$$h' - h = 0 \ldots\ldots\ldots\ldots\ldots (1).$$

Now
$$\frac{AK - h}{AG} = \frac{h - BM}{BG},$$

$$\therefore\ h = \frac{AK.BG + MB.AG}{AB},$$

$$= \frac{r \sin(a + \theta).b + r \sin\theta.a}{a + b}, \text{ if } AC = r,\ AG = a,\ BG = b,$$

$$= \frac{r}{a + b}\{b \sin(a + \theta) + a \sin\theta\};$$

similarly, $h' = \dfrac{r}{a + b}[b \sin\{(a + \theta) - \delta\theta\} + a \sin(\theta - \delta\theta)]$,

therefore $h' - h = \dfrac{r}{a + b}\{- b \cos(a + \theta).\delta\theta - a \cos\theta.\delta\theta\}$,

$$= 0 \text{ from } (1);$$

therefore since $\delta\theta$ is not zero,

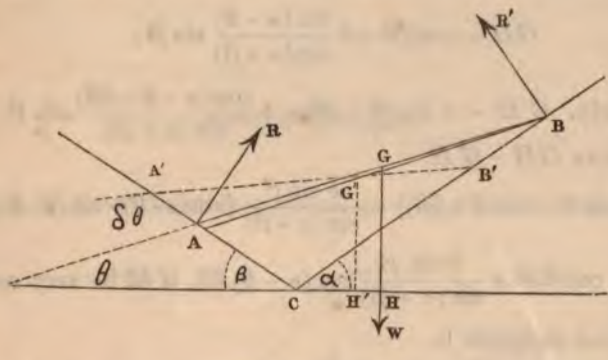
$$b \cos(a + \theta) + a \cos\theta = 0,$$

$$\tan\theta = \frac{a + b \cos a}{b \sin a},$$

which gives the value of θ required.

(3). A heavy beam AB rests with its extremities upon two smooth planes CA, CB inclined at angles β and a to the horizon, find its inclination to the horizon.

Let it equal θ; and suppose the beam to receive an inde-



finitely small displacement by sliding its ends A and B along the planes into the positions A' and B', its new inclination will be $\theta + \delta\theta$, where $\delta\theta$ is indefinitely small.

The external forces acting upon the beam are the resistances R and R' at A and B normal to the planes, and the weight W acting at G the centre of gravity of the beam: by the above displacement W alone has received any virtual velocity for A and B, both have moved in a line perpendicular to the direction of the forces applied to them.

To find the displacement of G in a vertical direction; let H be the point where the direction of W meets the horizontal line through C, H' the new position of H, corresponding to G' the new position of G; then $GH - G'H'$ is the virtual velocity of W, and we must have by the principle of virtual velocities,

$$W(GH - GH') = 0,$$

or

$$GH - GH' = 0 \ldots\ldots\ldots\ldots (1).$$

Now $$GH = AG \sin\theta + AC \sin\beta \ldots\ldots\ldots (2),$$

and from triangle ABC,

$$\frac{AC}{AB} = \frac{\sin ABC}{\sin ACB} = \frac{\sin(\alpha - \theta)}{\sin(\alpha + \beta)};$$

put $AG = a$, $AB = b$, both are known quantities, then by substitution above in equation (2),

$$GH = a \sin \theta + b \frac{\sin(\alpha - \theta)}{\sin(\alpha + \beta)} \sin \beta;$$

similarly, $G'H' = a \sin(\theta + \delta\theta) + b \frac{\sin(\alpha - \theta - \delta\theta)}{\sin(\alpha + \beta)} \sin \beta$,
therefore $GH - G'H'$

$$= a\{\sin \theta - \sin(\theta + \delta\theta)\} + \frac{b \sin \beta}{\sin(\alpha + \beta)}\{\sin(\alpha - \theta) - \sin(\alpha - \theta - \delta\theta)\}$$

$$= -a \cos \theta . \delta\theta + \frac{b \sin \beta}{\sin(\alpha + \beta)} \cos(\alpha - \theta)\, \delta\theta, \text{ if } \delta\theta \text{ be very small,}$$

$= 0$ from equation 1;

therefore, since $\delta\theta$ is not absolutely zero,

$$-a \cos \theta + \frac{b \sin \beta}{\sin(\alpha + \beta)} \cos(\alpha - \theta) = 0;$$

therefore $$\tan \theta = \frac{a \sin(\alpha + \beta) - b \cos \alpha \sin \beta}{b \sin \alpha \sin \beta},$$

which gives the required position of equilibrium.

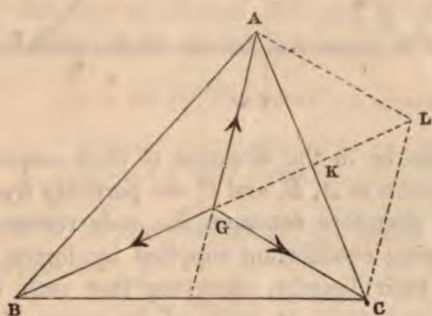
If $b = 2a$, or the beam be uniform, this becomes

$$\tan \theta = \frac{\sin(\alpha - \beta)}{2 \sin \alpha \sin \beta},$$

$\theta = 0$ when $\alpha = \beta$, *i.e.* the beam will then lie in a horizontal position.

General Examples.

(1). If three forces represented in magnitude and direction



by three lines GA, GB, GC, keep the body G at rest, then G will be in the centre of gravity of the triangle formed by joining the extremities of the lines.

The annexed figure represents the system. Complete the parallelogram $AGCL$, and draw the diagonal GL; it necessarily bisects AC the other diagonal in some point K.

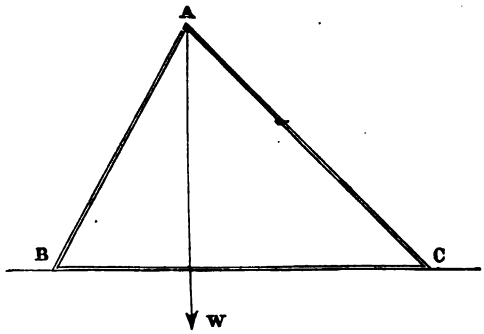
Now GL represents the resultant of the two forces represented by GA, GC; it must therefore be in the same straight line with BG; hence we see that BG produced bisects AC.

Similarly AG produced bisects BC.

Therefore G, the intersection of these two lines, is the centre of gravity of the triangle ABC.

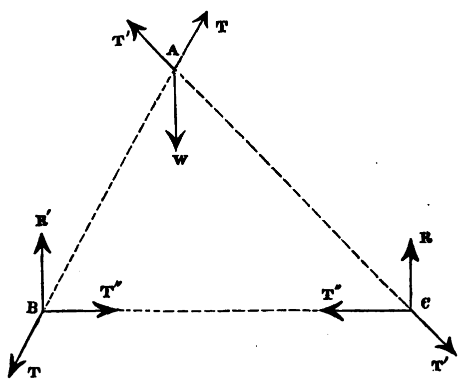
(2). Two rods without weight are jointed together at one extremity, the others are connected by a string of given length; if this triangle be placed upon a horizontal smooth plane, and a given weight be hung to the vertex, find the tension of the string.

Let AB, AC be the rods jointed at A, BC the string. The forces exerted upon the points A, B, and C by the rods and



the string must be in the direction of their respective lengths, because the joints at A, B, and C, are perfectly free. (Art. 9, γ.)

We may therefore conceive the rods removed, and their effect in keeping equilibrium supplied by forces acting in the direction of their lengths, observing that each rod or string exerts the same force at each of its extremities. On this supposition A is in equilibrium under the action of the force W acting vertically downwards, the tension T of the



rod AB in direction of BA produced, and tension T' of AC in direction of CA produced.

We have therefore

$$\frac{T}{W} = \frac{\sin T'AW}{\sin T'AT} =$$

$$= \frac{\sin WAC}{\sin BAC} = \frac{\cos C}{\sin A} \text{ .... (1)},$$

$$\frac{T'}{W} = \frac{\sin TAW}{\sin T'AT} = \frac{\cos B}{\sin A} \text{ .... (2)}.$$

Again, C is kept in equilibrium by T' the tension of the rod AC, T'' the tension of cord BC, and R acting vertically upwards the reaction of the smooth horizontal plane. Hence

$$\frac{T''}{T'} = \sin RCT' = \cos C \text{.............(3)},$$

$$\frac{R}{T'} = \sin T''CT' = \sin C \text{.............(4)}.$$

Similarly, if R' be the reaction of the plane at B, we get for equilibrium of B,

$$\frac{T''}{T} = \cos B \text{....................(5)},$$

$$\frac{R'}{T} = \sin B \text{....................(6)}.$$

Equations (1), (2), (3), (4), (5), (6), are *only five* independent equations, because (5) may be obtained from (1), (2), and (3). They determine T, T', T'', R and R'.

Multiplying (3) with (2), we have

$$\frac{T''}{W} = \frac{\cos B \cos C}{\sin A},$$

which gives the tension required, for A, B, and C are known when the lengths of the rods and string are known.

It is worth observing, that since from (2) and (4)

$$\frac{R}{W} = \frac{\cos B \sin C}{\sin A},$$

and from (1) and (6), $\dfrac{R'}{W} = \dfrac{\sin B \cos C}{\sin A},$

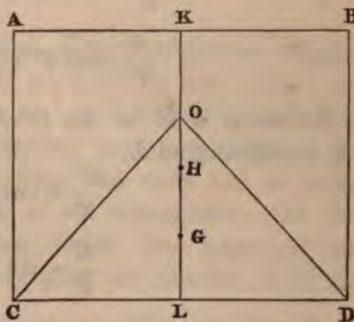
therefore $$\frac{R+R'}{W} = \frac{\sin(B+C)}{\sin A} = 1;$$

therefore the sum of the normal resistances of the plane is equal to the weight W, as might have been foreseen.

Also $$\frac{R}{R'} = \frac{\tan C}{\tan B}.$$

(3). It is required to cut from a rectangular board an isosceles triangle, having its base coincident with a side of the rectangle and such that the remaining portion has its centre of gravity at the vertex of the triangle.

Let $ABCD$ be the rectangle, KL a line parallel to AC bisecting AB and CD.



O the vertex of the required triangle COD, it is manifestly in KL.

H the middle point of KL is the centre of gravity of the whole rectangle; let G be the centre of gravity of COD, and O by hypothesis the centre of gravity of the portion $CABD$.

Then the weight of the whole rectangle acting at H is equivalent to the weight of COD acting at G, together with the weight of $CABD$ at O; therefore

$HG : HO$ :: weight of $CABD$: weight of COD,

:: rectangle AD − triangle COD : triangle COD,

$$:: KL - \tfrac{1}{2}OL : \tfrac{1}{2}OL;$$

therefore $$OG : HO :: KL : \tfrac{1}{2}OL,$$

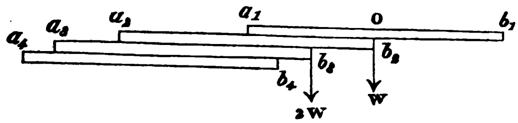
$$\tfrac{2}{3}OL : OL - \tfrac{1}{2}KL :: KL : \tfrac{1}{2}OL,$$

which gives the value of OL.

(4) A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it; if each projects as far as possible, prove that the distances

between the extremities of the successive cards will form an harmonic progression.

Let a_1b_1 be the farthest card, a_2b_2 its next lower neighbour, a_3b_3 its next, and so on; O_1O_2... the middle points of each, and



let w be the weight of each card, $2l$ its length. Since a_1b_1 projects as far as possible, its centre of gravity must be just supported by the extreme end of a_2b_2, therefore

$$b_2b_1 = l \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also, since a_2b_2 projects as far as possible, the resultant of its own weight w at O_2 and the pressure of the upper card equal w at b_2, *i.e.* a force $2w$ must pass through b_3; therefore

$$b_2b_3 = \tfrac{1}{2}O_2b = \tfrac{1}{2}l \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Again, considering the card a_3b_3, its weight w at O_3 together with $2w$ the weights of the upper cards acting at b_3, must have their resultant $3w$ passing through b_4; therefore

$$b_4b_3 : b_3O_3 :: w : 3w,$$

or

$$b_3b_4 = \tfrac{1}{3}b_3O_3,$$

$$= \tfrac{1}{3}l \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

In a similar way it could be shewn that

$$b_{n+1}b_n = \frac{1}{n}l,$$

therefore the distances which each successive card projects over its neighbour are

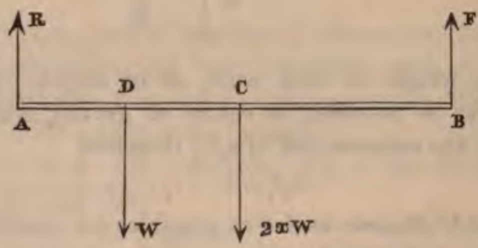
$$l, \tfrac{1}{2}l, \tfrac{1}{3}l \ldots \frac{1}{n}l:$$

as the reciprocals of these are in arithmetic, they must themselves be in harmonic progression.

(5). On a lever of uniform density, every inch weighing w oz. a weight of W oz. is suspended at a given distance from the

fulcrum, which is placed at one extremity, what must be the length of the lever, so that the whole may be supported by the least possible power acting in an opposite direction at the other extremity?

Let AB be the lever whose length $2x$ is required, A the fulcrum, D the given point where W is hung, C the middle



point of the lever, and therefore the point where its weight $2xw$ acts, $AD = a$, $AC = x$, F the force acting upwards at B. The lever AB is kept in equilibrium by F acting upwards at B, $2xw$ and W acting downwards at C and D respectively, and the resistance of the hinge at A. Hence the resultant of the first three forces must pass through the hinge A, therefore

$$AD.W + AC.2xw - AB.F = 0,$$

or $$aW + 2x^2w - 2x.F = 0 \ldots\ldots\ldots\ldots (1).$$

Equation (1) gives a relation between x and F: this may be put into a more convenient form, in order to discover the value of x corresponding to the least possible value of F, for

$$x^2 - \frac{F}{w}x = -\frac{aW}{2w};$$

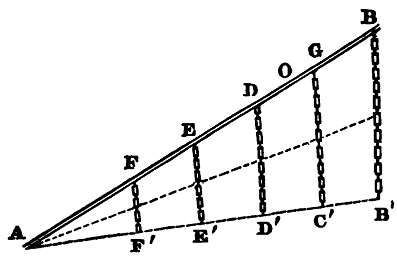
adding $\frac{F^2}{4w^2}$ to both sides of the equation, and extracting the root,

we have $$x - \frac{F}{2w} = \pm\frac{\sqrt{(F^2 - 2awW)}}{2w}.$$

From this it is easy to see that F cannot be less than $\sqrt{(2awW)}$; the value of x corresponding to the least value of F is $\frac{F}{2w}$.

(6). Five pieces of an uniform chain are hung at equidistant points along a rigid rod without weight, and their lower ends are in a straight line passing through one end of the rod: find the centre of gravity of the system.

Let AB be the rod; F, E, D, C, B the points at which the chains are hung, distant from each other by the same length b.



F', E', &c. the lower extremities of the chains. Since the lower end of each chain lies in the same straight line through A, its length must be proportionate to its distance from B. Hence, if

$$BB' = l,$$

then

$$CC' = \frac{AC}{AB}\,l,$$

$$DD' = \frac{AD}{AB}\,l,$$

&c. . . . .

If w be the weight of a unit's length of the chain, then the weight of

$$BB' = wl,$$

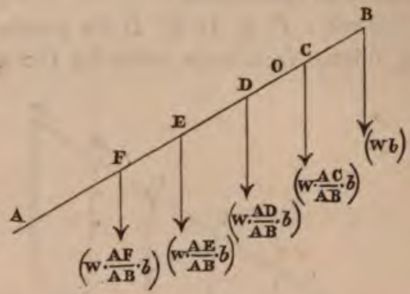
$$CC' = w\,\frac{AC}{AB}\,l,$$

$$DD' = w\,\frac{AD}{AB}\,l,$$

$$EE' = w\,\frac{AE}{AB}\,l,$$

$$FF' = w\,\frac{AF}{AB}\,l.$$

Now since the chains are perfectly flexible, they must always hang vertical, and their weights may be supposed to be applied at the points where they are attached, *i.e.* at F, E, D, C, B.



Hence if O be the centre of gravity of the system, it must lie in AB, and we must have

$$AO = \frac{wl.AB + \left(w\frac{AC}{AB}l\right)AC + \left(w\frac{AD}{AB}l\right)AD + \left(w\frac{AE}{AB}l\right)AE + \left(w\frac{AF}{AB}l\right)AF}{wl + w\frac{AC}{AB}l + w\frac{AD}{AB}l + w\frac{AE}{AB}l + w\frac{AF}{AB}l},$$

or $AO = \dfrac{AB^2 + AC^2 + AD^2 + AE^2 + AF^2}{AB + AC + AD + AE + AF}$.

If we put $AB = a$, this becomes

$$AO = \frac{a^2 + (a-b)^2 + (a-2b)^2 + (a-3b)^2 + (a-4b)^2}{5a - 10b},$$

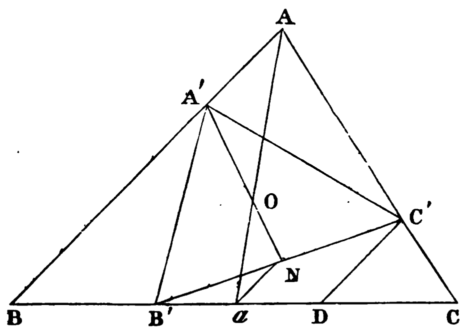
$$= \frac{a^2 - 4ab + 6b^2}{a - 2b}.$$

If BB', CC', &c. had been parallel rods rigidly attached to AB, it is manifest that the middle point of each would have been in the line AK joining A and the middle point K of BB', a line fixed with reference to AB and the rods.

In this case the weights of the rods necessarily acting at their middle points would not have their directions passing through the points of attachment, except when AB was so placed that the rods were vertical. Our supposition, therefore, of their acting at $BCDEF$ would not always be true; but now

the middle points themselves occupy fixed positions in the line AK, hence the centre of gravity of the system lies in this line, and may be found as before when it lay in the line AB.

(7). Let ABC be any triangle, $A'B'C'$ the simultaneous positions of three equal particles moving along AB, BC, CA, respectively, with such uniform velocities that A' describes AB,



B' describes BC, and C' describes CA, in the same time, their starting points being A, B, and C, respectively. The centre of gravity of $A'B'C'$ remains fixed.

Complete the triangle $A'B'C'$ and draw $C'D$ parallel to AB, Aa bisecting BC, $A'N$ bisecting $B'C'$ and cutting Aa in O; join aN; because DC' is parallel to BA,

$$\therefore DC : BC :: CC' : CA,$$
$$:: BB' : BC,$$
$$\therefore DC = BB',$$

and a bisects $B'D$; therefore aN is parallel to DC' and therefore to BA. Hence the triangles AOA', aON are similar, and

$$AO : Oa :: A'O : ON :: AA' : aN \ldots\ldots\ldots (1).$$

Now $aN = \frac{1}{2} DC'$, and

$$DC' : AB :: DC : BC,$$
$$:: BB' : BC,$$
$$:: AA' : AB,$$
$$\therefore DC' = AA',$$
$$\therefore aN = \tfrac{1}{2} AA'.$$

Therefore, from (1), $aE=\frac{1}{3}AE$, and $BD=\frac{1}{3}AB$, therefore G is the centre of gravity of both $A'B'C'$ and ABC, a constant point. Q.E.D.

For the following very neat proof of this proposition I am indebted to a friend.

If the particle A' be divided into two parts having to each other the ratio that AA' has to BB', and the first be placed at B, the second at A, the centre of gravity of the two parts will still remain at A'.

If the same thing be done with the particles B' and C', the centres of gravity of each pair of parts will be at B' and C' respectively, as before, and therefore the centre of gravity of the new system is the same as it would have been had the particles remained undivided at A', B', and C'. But it is evident that this arrangement leaves a mass equal to each particle at the points A, B, and C, as at first. Hence the centre of gravity of the whole system must always remain undisturbed.

(9). An inextensible string binds together two smooth cylinders, whose radii are given; find the ratio of the pressure between the cylinders to the tension of the string.

The only forces which act upon either of the cylinders are the tension of the string passing round it, and the pressure of the other cylinder at the point of contact. Since the cylinders are smooth the tension of the string will be uniform throughout its length; let it equal T, and let the mutual reaction of the spheres be represented by R.

The annexed figure shews the section of the two cylinders

by a plane through the cord, perpendicular to the axes of the cylinders; $O_1 O_2$ are the centres of the circular sections of the cylinders, $P_1 Q_1$ are the points in circumference of the first where the contact of the string ceases, $P_2 Q_2$ similar points in the second. It is clear that, after the equilibrium is established, no disturbance will be produced by nailing or in any way fixing the cord to the smaller cylinder at the points $P_2 Q_2$, which consideration shews that the tension of the cords may be supposed to be applied immediately at those points in direction making right angles with the radius. The cylinder O_2 is kept at rest by these two forces and the reaction R of the other sphere which acts in the line joining $O_1 O_2$; the directions of these three forces therefore meet in a point K upon $O_1 O_2$ produced, and if K be supposed rigidly connected with the sphere, the forces themselves may be considered to act there. Art. (15).

Hence, we have

$$\frac{R}{T} = \frac{\sin P_2 K Q_2}{\sin TKR}$$

$$= \frac{\sin P_2 K Q_2}{\sin \frac{P_2 K Q_2}{2}}.$$

But drawing $O_2 N$ parallel to $P_2 P_1$, and calling the radii of the cylinders r_1 and r_2 respectively,

$$\sin O_1 O_2 N = \frac{r_1 - r_2}{r_1 + r_2} = \sin \frac{P_2 K Q_2}{2};$$

therefore $$\frac{2\sqrt{(r_1 r_2)}}{r_1 + r_2} = \cos \frac{P_2 K Q_2}{2},$$

and $$\sin P_2 K Q_2 = \frac{4\sqrt{(r_1 r_2)}(r_1 - r_2)}{(r_1 + r_2)^2};$$

therefore $$\frac{R}{T} = \frac{4\sqrt{(r_1 r_2)}}{r_1 + r_2}.$$

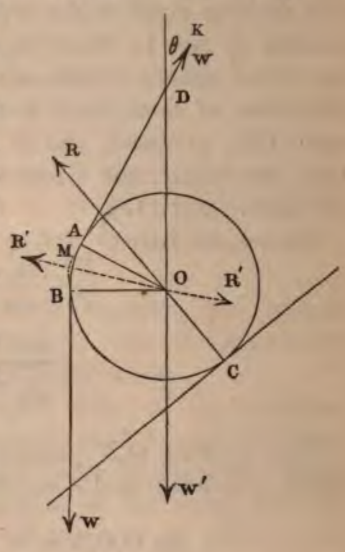
(9). A smooth cylinder is supported upon an inclined plain with its axis horizontal, by means of a string passing over the upper surface of the cylinder, which has one end

attached to a fixed point and the other to a weight W, which hangs freely.

If α be the inclination of the plane to the horizon, and θ the inclination of the string to the vertical, the weight of the cylinder is

$$= 2W \frac{\sin\frac{\theta}{2}\cos\left(\alpha + \frac{\theta}{2}\right)}{\sin\alpha}.$$

The annexed figure represents the system. K is the point at which the string is fixed, AB the portion of the cylinder with which it is in contact. Since the cylinder is perfectly smooth, the tension of the string is the same throughout its length $= W$. It is manifest that equilibrium will be in no way effected if we suppose the string to be nailed or otherwise fastened to the sphere at A and B: but in this case these points would become the points of application of the tensions W of the string.



Hence we may conceive the cylinder to be in equilibrium under the action of the following forces,

W acting at A along AK making angle θ with vertical,
W .... B .... BW vertical,
W' the weight of sphere acting at O the centre,
R the reaction of plan at C, also passing through O.

The first two may be conceived to act at the point M where their directions meet, Art. (15); and their resultant acting at this point must be equal and opposite to the resultant of R and W' at O, call it R'; its direction OM clearly bisects the angle AOB, which equals θ; also $ROD = COW' = \alpha$.

Now because R' is the resultant of W' and R at O, we must have

$$W' : R' :: \sin ROR' : \sin ROW'$$
$$:: \sin ROM : \sin ROD$$
$$:: \cos(BOM + DOR) : \sin ROD,$$

i.e. $$W' : R' :: \cos\left(\frac{\theta}{2} + \alpha\right) : \sin \alpha \ldots\ldots\ldots (1).$$

Again, because R' is the resultant of W and W at M,

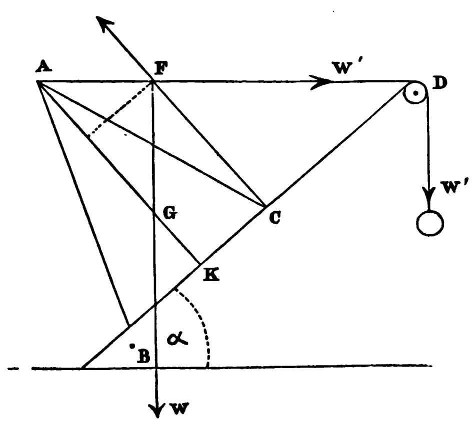
$$R' : W :: \sin KMW : \sin R'MK$$
$$:: \sin AOB : \sin AMO$$
$$:: \sin \theta : \cos \frac{\theta}{2} \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

therefore, compounding (1) and (2),

$$W' : W :: 2 \sin \frac{\theta}{2} \cos\left(\frac{\theta}{2} + \alpha\right) : \sin \alpha,$$

or $$W' = 2W \frac{\sin \frac{\theta}{2} \cos\left(\frac{\theta}{2} + \alpha\right)}{\sin \alpha}.$$

(10). A cone of given weight W is placed with its base on a smooth inclined plane and supported by a weight W'



which hangs by a string fastened to the cone and passing over a pulley in the inclined plane at the same height as the vertex. Find the angle of the cone when the ratio of the weights is such that a small increase of W' would cause the cone to turn about the highest point of the base as well as slide.

Let BCD be the plane inclined at an angle α to the horizon, ABC the cone, G its centre of gravity.

Let the horizontal direction of W' applied at A, and the vertical direction of W, at G, meet in F; then the resultant of W and W' must pass through F: since for the equilibrium of the cone this must be counteracted by the reaction of the smooth plane, it is sufficient that its direction be perpendicular to the plane, and meet it in some point between B and C. Art. (32).

But by the question the slightest addition to W' would make the cone both slide and turn about C, that is, make the resultant of W and W' oblique to the plane, and meet it beyond the point C; therefore its direction must now be a normal to the plane at C, or FC must be at right angles to BCD.

Let AGK be the axis of the cone, $KAC = \theta$.

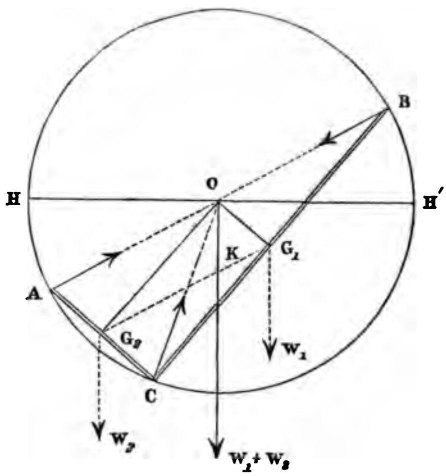
Then
$$\frac{AF}{KC} = \frac{1}{\cos\alpha},$$
$$\frac{KC}{AK} = \tan\theta,$$
$$\frac{AG}{AF} = \frac{1}{\sin\alpha};$$

multiplying these together,
$$\frac{AG}{AK} = \frac{\tan\theta}{\sin\alpha\cos\alpha}.$$

But because G is the centre of gravity of the cone,
$$AG = \tfrac{3}{4}AK,$$
therefore
$$\tan\theta = \tfrac{3}{4}\sin\alpha\cos\alpha$$
$$= \tfrac{3}{8}\sin 2\alpha.$$

(11). Two uniform heavy beams are rigidly fixed together in such a manner as to form the two sides of a right-angled triangle: the system rests in a vertical hoop with all the angles in contact with the rim; shew that the inclination of the beams to the horizon is independent of their length.

Let O be the centre of the hoop AC, CB the beams making with each other a right angle at C. Draw OG_1, OG_2 perpendicular to BC, AC respectively; then G_1G_2 are the



middle points of BC and AC, and therefore the points where their weights W_1, W_2 act: the centre of gravity of the whole system will consequently be a point K in the line joining G_1 and G_2 such that

$$G_1K : G_2K :: W_2 : W_1 \ldots\ldots\ldots\ldots (1).$$

Now the beams are in equilibrium under the action of the resistances of the hoop at A, B, and C, and their own weight $W_1 + W_2$ acting vertically downwards at K.

But the directions of each of the three resistances is normal to the hoop, and therefore passes through O; therefore the direction of $W_1 + W_2$ must also pass through O; in other words, the vertical line through O must cut G_1, G_2 in K.

Draw HOH' horizontal, and call HOG_2, θ; then

$$\frac{OG_1}{G_1K} = \frac{\sin K}{\sin KOG_1},$$

$$\frac{OG_2}{G_2K} = \frac{\sin K}{\sin KOG_2};$$

therefore $$\frac{OG_1}{OG_2}\,\frac{G_2K}{G_1K} = \frac{\sin KOG_2}{\sin KOG_1}\ldots\ldots\ldots\ldots(2),$$

and from the above,

$$\frac{G_2K}{G_1K} = \frac{W_1}{W_2} = \frac{BC}{AC},$$

because the beams are uniform; and

$$\frac{OG_1}{OG_2} = \frac{\frac{1}{2}AC}{\frac{1}{2}BC} = \frac{AC}{BC};$$

therefore, by substitution in (2), we find

$$1 = \cot\theta;$$

therefore $$\theta = 45^\circ,$$

which is independent of the length of the beams.

(12). A square is placed with its plane vertical between two smooth pegs which are in the same horizontal line; shew that it will be in equilibrium when the inclination of one of its edges to the horizon

$$= \frac{1}{2}\sin^{-1}\frac{a^2 - c^2}{c^2},$$

$2a$ being the side of the square, and c the distance between the pegs.

Interpret the result when $a = c$.

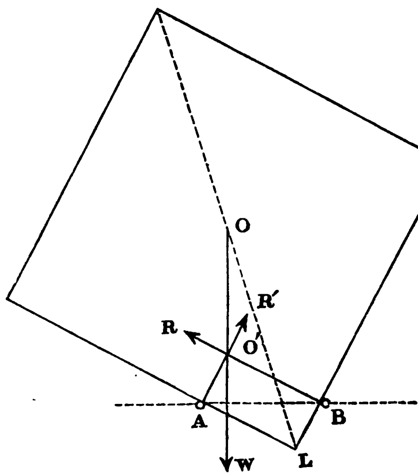
Let A and B be the two pegs, AB is by question horizontal.

Let O be centre, and therefore centre of gravity of the square.

The square is in equilibrium under the action of the resistances of the pegs A and B normal to its sides, and its own weight, W acting vertically downwards at O.

The resistances meet at right angles in some point O', hence W, which counteracts their resultant, must also pass through O',

or in other words, OO' must be vertical and therefore perpendicular to AB; let it meet AB in K.



Draw OM* perpendicular to AO', produced, and therefore parallel to the side AL of the square.

The triangle $O'MO$ is a right-angled triangle similar to ALB, therefore

$$\frac{O'M}{OM} = \frac{AL}{BL}.$$

Put $LAB = \theta$, so that $AL = c\cos\theta$, $BL = c\sin\theta$; then by substitution,

$$\frac{a - c\sin\theta}{a - c\cos\theta} = \frac{\cos\theta}{\sin\theta},$$

therefore $\quad a(\sin\theta - \cos\theta) = c(\sin^2\theta - \cos^2\theta)$,

or $\quad \{a - c(\sin\theta + \cos\theta)\}(\sin\theta - \cos\theta) = 0.$

Taking the second factor, we get

$$\tan\theta = 1;$$

hence there will always be equilibrium when both sides are equally inclined to the horizon.

* This construction has been erroneously omitted in the figure.

Taking the first factor we get, by squaring,

$$\left(\sin\theta + \cos\theta\right)^2 = \frac{a^2}{c^2},$$

or $$1 + 2\sin\theta\cos\theta = \frac{a^2}{c^2};$$

therefore $$\sin 2\theta = \frac{a^2}{c^2} - 1,$$

$$\theta = \frac{1}{2}\sin^{-1}\frac{a^2 - c^2}{c^2}.$$

(13). A cylinder whose height is $2a$ stands with the centre of its base upon the top of a rough sphere whose radius is r; after being rolled with its base in contact with the sphere through an angle θ, it is left to itself and is observed to roll back to its first position; what is the greatest value of which θ is capable?

The annexed figure represents a vertical section of the cylinder and sphere through the axis of the former: A is the original point of contact, C the centre of gravity of the cylinder vertical above it, $AC = a$; O is centre of sphere, $AO = r$.

Let B be the new point of contact, A' C' the new positions of A and C, when the cylinder has rolled upon the sphere through the angle θ, therefore $AOB = \theta$; let also the direction of the weight W at C' meet the base $A'B$ in K.

Since the sphere and cylinder are both supposed to be quite rough, and the only forces acting upon the cylinder are the reaction of the sphere at B and the weight W at C, motion can only commence by W making the cylinder roll about B, as if B were fixed; and it is clear that this rolling will take place towards or from A, according as K lies to the right or left of B, *i.e.* according as $A'K$ is less or greater than $A'B$, or according as $A'K$ is less or greater than AB.

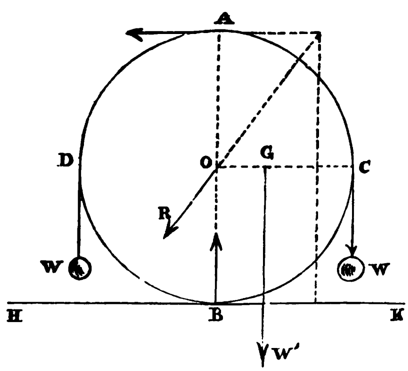
Now $$A'K = A'C' \tan\theta$$
$$= a \tan\theta,$$
and $$AB = r\theta.$$
Therefore the cylinder will roll towards A or from A according as
$$a \tan\theta \text{ is less or greater than } r\theta.$$

Hence, in the question, the greatest value of θ is given by the equation
$$a \tan\theta = r\theta.$$

If θ actually had the value given by this equation, the direction of W would pass through B, and there would be equilibrium: the equilibrium in this position would manifestly be unstable, because for every other value of θ, whether greater or less, the cylinder would by the preceding reasoning roll away.

(14). A smooth body in the form of a sphere is divided into hemispheres and placed with the plane of division vertical upon a smooth horizontal plane; a string loaded at its extremities with two equal weights hangs upon the sphere passing over its highest point and cutting the plane of division at right angles: find the least weight which will preserve equilibrium.

Let $ABCD$ be the sphere, HBK the plane on which it



rests, O its centre, AOB the line in which the plane of division meets that of the paper. Consider the equilibrium of the

hemisphere $ACBO$ alone. Since the string AC rests in contact with the $\frac{1}{2}$ sphere, the conditions of equilibrium will not be at all altered by supposing it glued to it; but in this case it is clear that the $\frac{1}{2}$ sphere is kept in equilibrium by the action of its own weight W' acting vertically at G its centre of gravity,

.... horizontal tension of string at $A = W$,

.... vertical tension of the string at $C = W$,

.... vertical reaction of plane at B,

and horizontal reaction of other $\frac{1}{2}$ sphere somewhere in AB.

Since the two last can be exerted to any required amount there will be always equilibrium maintained if the resultant of the three first meet the vertical line somewhere between A and B.

Now as W is diminished, the direction of this resultant will evidently be inclined towards the vertical force $W + W'$ away from the horizontal W, *i.e.* will be depressed towards B, and W will be least possible consistent with equilibrium when it passes through B: but then the sum of the moments of these three forces must vanish about B (Art. 28), or

$$OG \times W' + OC \times W - BA \times W = 0;$$

therefore calling the radius of sphere a, since $OG = \frac{3}{8} a$,

$$\frac{3}{8} a W' - a W = 0,$$

$$W = \frac{3}{8} W'.$$

(15). A heavy uniform beam has one extremity in a smooth spherical bowl, the other projects over the edge; the bowl is placed upon a smooth horizontal plane: required the position of equilibrium of the system.

Let the figure represent the beam and bowl in equilibrium,

AC the beam,

O the centre of the bowl,

$KADB$ the bowl, D the point of contact with the horizontal plane.

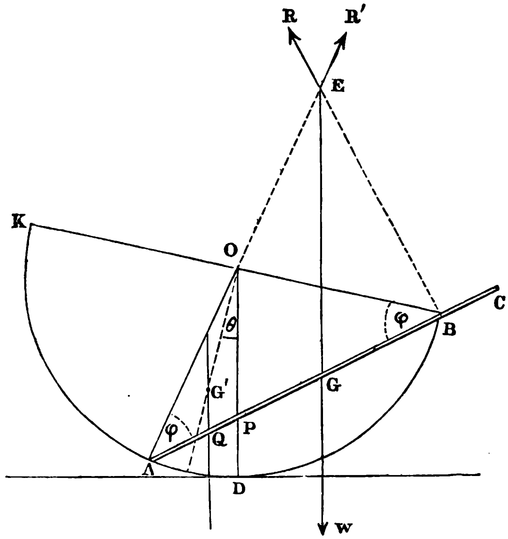
Then the beam is held in equilibrium by the forces

R' the reaction at A in direction AO,

R .... B .... perpendicular to ABC,

W the weight of beam at G vertically downwards.

Now R and R' meet at E in the circumference KDB pro-



duced, hence W must also pass through E, or EG must be vertical and therefore parallel to OPD;

therefore $PG : AG :: AE : AO$, or if $AC = 2b$ and $AO = a$,

$$PG : AG :: 1 : 2,$$

$$PG = \tfrac{1}{2}b \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now let G' be centre of gravity of the bowl, and Q the point where the vertical through G' meets ABC; also let

$$OG' = c, \quad DOG' = \theta \; OAB = \phi = OAB,$$

therefore $$QP = OG' \frac{\sin G'OP}{\sin OPB} = c \frac{\sin\theta}{\cos(\phi - \theta)} \ldots\ldots (2).$$

The whole system of the bowl and beam together is kept in equilibrium by the weight w of bowl at G', the weight W of beam at G, and the reaction of horizontal plane at D; therefore the resultant of the first two must pass through P vertically, or we must have

$$W.GP = w.QP,$$

i.e.
$$W\frac{b}{2} = w\,\frac{c\sin\theta}{\cos(\phi-\theta)} \quad \text{.............} \quad (3),$$

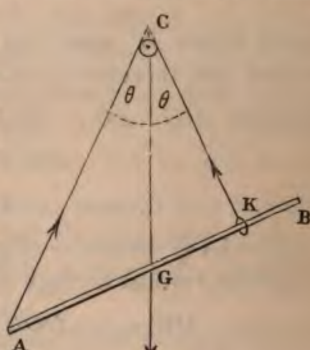
also
$$\frac{b}{2a} = \frac{\sin AEG}{\sin EGB} = \frac{\sin(OPB - OAB)}{\sin OPB}$$

$$= \frac{(\cos 2\phi - \theta)}{\cos(\phi - \theta)} \quad \text{.............} \quad (4).$$

Equations (3) and (4) will determine θ and ϕ, and therefore the positions of equilibrium.

(16). One end of a string is fixed to the extremity of a smooth uniform rod, and the other to a ring through which the rod passes, the string is then hung over a smooth peg. Determine the least length of string for which equilibrium is possible, and shew that the inclination of the rod to the vertical cannot be less than 45°.

Let AB be the rod suspended by the string ACK, which passes over the smooth peg C.



The rod is kept in equilibrium by three forces, the tension of the string at A, the tension of the string at K, and the weight of itself at G: the first two are equal and their directions meet in C, therefore the direction of the third, *i.e.* the vertical line through G, must pass through C and bisect the angle ACK: put $GCK = \theta$; CKG is a right angle because there is no friction between the ring K and the rod.

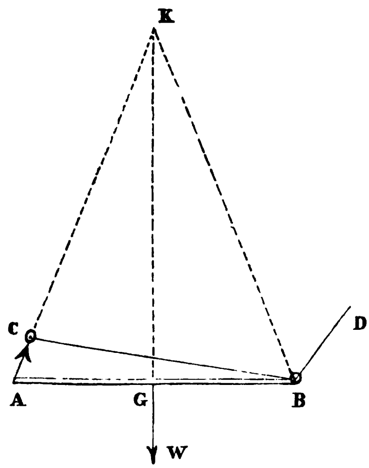
Then from triangle CGA,

$$CG = AG\,\frac{\cos 2\theta}{\sin\theta} = AG\left(\frac{1}{\sin\theta} - 2\sin\theta\right).$$

From this we see that θ increases, and therefore CGB diminishes as CG diminishes, and the least value of CG is

evidently when $\sin\theta = \frac{1}{\sqrt{2}}$ or $\theta = \frac{\pi}{4}$, therefore the least value of CGB is $\frac{\pi}{4}$. Also it appears that the least length of string required corresponds to this angle, and therefore equals AG.

(17). A uniform rod AB is supported as follows: a string



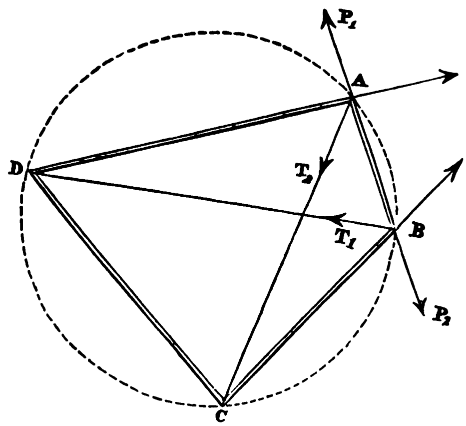
is tied to A, passes through a fixed external ring C, then through a ring in the rod at B, and is finally fixed to an external point D: shew that if AB is horizontal, the angle DBC is 120°.

The beam is kept in equilibrium by the tension of the string AC at A, the resultant of the two tensions of BC and BD at B, and its own weight W at G the middle point of AB.

Let AC and the vertical through G meet in K; the direction of the resultant of tensions at B must also pass through K and must evidently equal the tensions of AC at A, from the symmetry of the figure; but this force is the resultant of two forces which are also equal the tension of AC, and whose directions are BC and BD, therefore it can be easily seen that it must make an angle equal 60° with both of them: hence the whole angle $CBD = 120°$.

(18). Four rods jointed at their extremities form a quadrilateral which may be inscribed in a circle: if they be kept in their positions by two strings joining the opposite angular points, the tension of each string is inversely proportional to its length.

Let $ABCD$ be the quadrilateral AC, BD the strings which



keep it in shape: since the joints at $ABCD$ are supposed to be perfectly free, the tensions of the rods and the strings will be in the direction of their lengths: let the tensions of BD and AC be T_1 and T_2 respectively, and that of AB be P_1.

Then A is kept in equilibrium by the tension of the rods AB and AD, and the string AC, in the direction of their lengths respectively,

therefore $$\frac{T_2}{P_1} = \frac{\sin A}{\sin DAC}.$$

Similarly, from equilibrium of B,

$$\frac{P_1}{T_1} = \frac{\sin DBC}{\sin B},$$

therefore $$\frac{T_2}{T_1} = \frac{\sin A}{\sin B}, \quad \because DAC = DBC.$$

Now from triangle DAB,

$$\frac{\sin A}{\sin ADB} = \frac{BD}{AB},$$

and from triangle BAC,

$$\frac{\sin B}{\sin ACB} = \frac{AC}{AB};$$

therefore by division, since $ACB = ADB$, we have

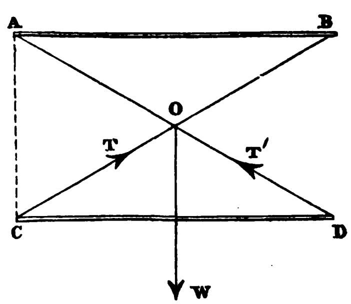
$$\frac{\sin A}{\sin B} = \frac{BD}{AC} = \frac{T_2}{T_1},$$

which proves the proposition.

(19). A quadrilateral formed by four rods jointed at the ends is in equilibrium under the action of two equal and opposite forces applied at points in two opposite sides. Shew that the direction of these forces passes through the point of intersection of the other two sides produced; these being supposed not parallel.

This follows from the consideration that either of the sides to which the force is applied, is kept in equilibrium by this force together with the pressure of the two adjacent sides, in the direction of their lengths, upon its ends.

(20). Two equal heavy beams AB, CD are connected



diagonally by similar and equal elastic strings AD, BC; determine the position of equilibrium when AB is held horizontally. Shew also that if the natural length of each string equal AB,

and the elasticity be such that the weight of AB would stretch the string to three times its natural length, then

$$\frac{1}{AB} = \frac{1}{BC} + \frac{1}{AC}.$$

Let T be the tension of each of the two strings when the system is in equilibrium, and let the strings meet in O; then W being the weight of either of the beams, we have

$$\begin{aligned}\frac{T}{W} &= \frac{\sin AOW}{\sin AOB} \\ &= \frac{\cos ABO}{\sin 2ABO} = \frac{1}{2\sin ABO} \\ &= \frac{1}{2}\frac{BC}{AC} \\ &= \frac{1}{2}\frac{BC}{\sqrt{(BC^2 - AB^2)}}.\end{aligned}$$

Now the tension of an elastic string is proportional to the amount of its extension; hence, if AB were the original length of the string, we have

$$T = \lambda\,(BC - AB),$$

where λ is a constant for all values of BC; therefore by substitution

$$\frac{\lambda\,(BC - AB)}{W} = \frac{1}{2}\frac{BC}{\sqrt{(BC^2 - AB^2)}};$$

an equation which will serve to determine BC, and therefore the position of equilibrium.

The supposition of the second part of the question gives us

$$\lambda\, 2AB = W,$$

and therefore the above relation becomes

$$\begin{aligned}\frac{BC - AB}{AB} &= \frac{BC}{\sqrt{(BC^2 - AB^2)}} \\ &= \frac{BC}{AC};\end{aligned}$$

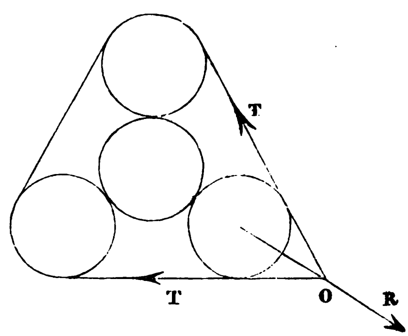
or

$$\frac{1}{AB} = \frac{1}{BC} + \frac{1}{AC}.$$

(21). Three equal cylindrical rods are placed symmetrically round a fourth one of the same radius, and the bundle is then surrounded by two equal elastic bands at equal distances from the two ends: if each band when unstretched would just pass round one rod, and a weight of one pound would just stretch it to twice its natural length, shew that it would require a force of 9 lbs. to extract the middle rod, the coefficient of friction being equal to $\frac{\pi}{6}$.

Since the arrangement of the rods and the bands is perfectly symmetrical, the conditions between the forces must be the same at both bands. Let the annexed figure represent a section of the rods by a plane passing through one of the bands; it is clear that the case is the same as that of four equal circular discs pressed together by an elastic band.

Now each of the exterior discs will be under the same



circumstances: it will be kept in equilibrium by the tension of two parts of the string and the reaction of the central disc. Call this reaction R and the tension T: these three forces will, by the symmetry of the figure, clearly meet in a point O, which is the angle of the equilateral triangle described about the discs; hence from the triangle of forces

$$R = 2T \cos 30^\circ = T\sqrt{3}.$$

If therefore μ be the coefficient of friction between the rods, the friction which would upon attempting to withdraw the

middle rod be called forth corresponding to the above value of R, would be μR

$$= \mu T\sqrt{3}.$$

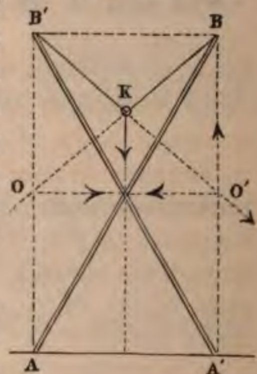
The same would be the case at the other five points of contact (three belonging to the other band); therefore the total amount of friction to be overcome $= 6\mu\sqrt{3}\,T = \pi\sqrt{3}\,T$ by the question.

Now if r be the radius of each cylinder, the original length of each band was $2\pi r$, the stretched length is $12r\cos 30° + 2\pi r$, therefore the amount of extension $= 12r\cos 30° = 6\sqrt{3}r$: but by question 1 lb. would produce an extension $= 2\pi r$; hence we have

$$\text{value of } T \text{ in lbs.} : 1 \text{ lb.} :: 6\sqrt{3} : 2\pi,$$

therefore $$\pi\sqrt{3}\,T = 9 \text{ lbs.}$$

(22). Two equal rods AB, $A'B'$ without weight, are connected at their middle points by a pin C which allows of free motion in a vertical plane, and their upper extremities are connected by a thread $B'KB$, which carries a weight hanging by a free ring K. Whatever be the length of the string K will always rest half way between the pin C and the horizontal line joining BB'.



It is quite evident that the whole system will arrange itself perfectly symmetrically, and that K will always be somewhere in the vertical line through C.

The beam $A'B'$ is in equilibrium under the action of the tension of the string $B'K$ at B', the resistance of the horizontal plane at A', in direction $A'B$ and the reaction of the other rod called forth at C: let $B'K$ and $A'B$ meet in O', then the direction of their resultant must be $O'C$. Of course, by similar reasoning, the resultant of the tension of BK and the resistance at A would have a direction OC symmetrical with $O'C$: now the reaction at C which counteracts this force must be equal and opposite to the reaction at C, which counteracts the re-

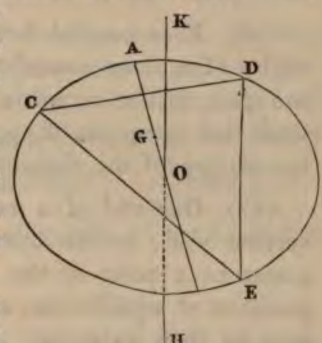
sultant force along OC; therefore OCO' must be a straight line, and as CO' is symmetrical with CO, it must be horizontal;

therefore $$A'O' = BO'.$$

Also, because $B'A'$ is bisected in C, and KC is parallel to OA', therefore $KC = \frac{1}{2}O'A' = \frac{1}{2}BO'$, which proves the proposition.

(23). The three corners of a triangle are kept on a circle by three rings capable of sliding along the circle, and the circle is inclined to the horizon at a given angle; find the positions of equilibrium.

Let O be the centre of the circle; KH a vertical line through it; AOB the line in which the plane of the circle is cut by a plane perpendicular to it through O and KO; CDE the triangle.



This is kept in equilibrium by the three reactions of the circle upon the rings at CD and E, and by its own weight acting vertically downwards at G.

The three first are perpendicular respectively both to the ring and the circle; they must each therefore lie in a plane which passes through the centre and the angle, and is perpendicular to the plane of the circle, *i.e.* they must each pass through a line drawn perpendicular to the circle at O. Also they must have a single resultant from what is said above, vertical, equal and opposite to the weight of the triangle, and this resultant must clearly pass through the line drawn perpendicular to the planeof the circle at O; therefore the vertical line through G the centre of gravity of the triangle must intersect the same perpendicular, or must lie in the plane of KOH and AOB, therefore G must lie in the line AOB: and this is the only condition necessary for equilibrium, consequently there are two positions such as are required of the triangle, one when G is above, and another when it is below the centre O in the line AOB. Q. E. D.

(24). A man carries a bundle at the end of a stick over his shoulder: as the portion of the stick between his shoulder and hand is shortened by the moving his hand, shew that the pressure on his shoulder is increased. Does this change increase the pressure of his feet upon the ground?

(25). Two weights support each other upon the two sides of a solid triangle by means of a string which passes over the vertex and connects them: compare the weights. What would be the effect upon equilibrium, first of adding the same weight to each, secondly of adding weights to them which are in the same proportion as themselves?

(26). Four parallel forces act in the same direction at the angles of a plane quadrilateral figure, and are inversely, pair and pair, to the segments of the diagonals nearest to them; shew that the point of application of their resultant lies at the intersection of the diagonals.

(27). One end of a uniform beam rests against a smooth vertical wall; to the other end a string is attached, which passes to a point in the wall where it is fixed. Find the positions of equilibrium, and investigate the conditions necessary for their existence.

(28). A uniform beam lies over a smooth peg and has one of its extremities attached by a string to a point vertically above the peg; find the conditions of equilibrium.

(29). Two weights are hung by different strings to the same point; one of these weights is a sphere of known size, the other rests far below it on account of its string being long: find the position of equilibrium.

(30). A heavy weight W lies upon a smooth table; a long string fixed to it passes over a smooth peg above the table, returns to the weight, passes over a smooth peg in it, and then going through a slit in the table carries a weight P; another string is also fixed to W, passes over the edge of the table, and then carries a weight Q. Find the position of equilibrium and the pressure on the table.

(31). A weight is suspended from the two ends of a straight lever, whose length is 5 feet, by strings whose lengths are 3 and 4 feet respectively; find the position of the fulcrum that the lever may rest in a horizontal position, first supposing the lever to be without weight, secondly to be heavy.

(32). A uniform rod is hung from a fixed point by means of cords attached to its extremities; shew that the tensions of the cords are as their lengths.

Is this true, first when the points of attachment of the cords are at equal distances, secondly at unequal distances from the extremities of the rod?

(33). Two equal weights P, P are attached to the ends of two strings which pass over the same smooth peg and have their other extremities attached to the ends of a beam AB (weight W) which rests thus suspended; shew that the inclination of the beam to the horizon $= \tan^{-1}\left(\frac{a-b}{a+b}\tan\alpha\right)$; a, b being the distances of the centre of gravity of the beam from its ends, and $\sin\alpha$ being $= \frac{W}{2P}$.

(34). A bent uniform rod is hung up by one extremity, find its position of equilibrium: find also the angle between the two arms in terms of their lengths when the lower one is horizontal.

(35). Find the centre of gravity of a frustrum of a right cone.

(36). A triangular lamina ABC whose weight is W is suspended by a string fastened at C, find the weight which must be attached at B, in order that in the position of equilibrium the vertical through C shall bisect the angle ACB.

(37). A cone whose height is four times the radius of its base is hung up by a string attached to a point in the circumference of the base; find the angle which its axis makes with the vertical.

(38). A uniform wheel having a heavy particle fixed at a given point in its circumference is placed upon a rough inclined plane; shew that if the weight of the particle be of a certain value, there is but one position of equilibrium, if it be greater there are two positions, if less there are none.

Is this independent of the coefficient of friction?

Investigate the nature of the stability of equilibrium in each case.

(39). A cylinder of given weight and radius stands upon a horizontal plane, a sphere of weight W' hangs by a string from the upper rim of the cylinder and touches its side; what must be its radius in order that the cylinder may be just on the point of upsetting?

(40). A sphere of known weight and dimensions is hung by a string of given length to the vertex of a cone and rests upon its surface; the cone is then placed upon a horizontal plane; if the string be now slightly lengthened the whole system will overset: find the relation between the weight, height, and vertical angle of the cone.

(41). Two heavy rings are made to slide uniformly along the sides of a triangle, starting together from the vertex and reaching the base at the same moment; shew that if a certain straight line in the triangle be fixed, the rings will be in equilibrium in any of their contemporaneous positions.

(42). A cubical box is half filled with water and placed upon a rough rectangular board so as to have the edges of its base parallel to those of the rectangle: if the board be slowly inclined to the horizon, determine whether the box will slide down or topple over.

(43). $BCDE$ is a square board; a string is fixed to a point A in a rough wall and to the corner B of the board; shew that the board will rest with its plane perpendicular to the wall and its side CD resting against it, if AC be not greater than μBC.

(44). A sphere is placed upon a horizontal plane, and a heavy beam capable of moving about a hinge at one extremity in the

same plane lies upon it; given the coefficient of friction between the beam and sphere; also that the sphere cannot slide but only roll along the plane: find the inclination of the beam to the horizon when the sphere is just about to roll away.

(45). A heavy beam lies over a peg with one extremity against a vertical wall; given the coefficient of friction between the beam and peg, and between the beam and wall; find the limiting positions of equilibrium.

(46). One end of a beam, whose weight is W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests against another smooth plane inclined at an angle α to the horizon; the string passing over a pulley at the top of the inclined plane hangs vertically supporting a weight P. Shew that the beam will rest in all positions if a certain relation hold between P, W, and α.

(47). A beam whose length is $2a$ has two beams each equal, a jointed one at each of its extremities; the three beams are then placed on the top of a sphere whose radius is $2a$, in such a way that the middle point of the middle beam rests on the highest point C of the sphere: shew that the pressure on the sphere at C is equal to $\frac{91}{100}$ of the whole weight of the three beams.

(48). Two equal smooth spheres connected by a string are laid upon the surface of a cylinder, the string being so short as not to touch the cylinder: determine the position of rest and the tension of the string.

(49). A weight Q is placed inside a smooth hemispherical cup, a string attached to it passes over the smooth edge of the cup and supports a weight P hanging vertically; the hemisphere rests with its curved surface in contact with a smooth horizontal plane: find the position of equilibrium. If the cup be without weight, how must the enunciation of the problem be altered in order that equilibrium may be possible?

(50). A series of equal weights are fixed at equal distances along a string which is suspended from two fixed points; shew that the successive portions of the string are inclined to

the horizon at angles whose tangents are in arithmetical progression.

(51). A light cord with one end attached to a fixed point passes over a pulley in the same horizontal line with the fixed point, and supports a weight hanging freely from the other extremity. A heavy weight being *fixed* upon the cord at different places successively between the fixed point and pulley, it is required to find the locus of its positions of equilibrium.

(52). An elliptical plane, in which A and B are the extremities of the major and minor axes and S the focus respectively, is acted upon by two forces applied at the point B, which are in directions of, and proportional to BA and BS: what point in the major axis must be fixed in order that motion may not ensue?

(53). A triangular board has one right angle, and is placed in a vertical plane with one side horizontal; a heavy string covers the hypothenuse and hangs down at the lower end; the upper end of the string is held by a force parallel to its length: find a point in the board which must be fixed in order that the base may remain horizontal.

(54). Three equal spheres lying in contact on a horizontal plane are held together by a string. A cube whose weight is $3W$ is placed with one of its diagonals vertical so that its lower sides touch the spheres: shew that the tension of the string is $W\sqrt{\frac{2}{3}}$.

(55). The piston rod of a locomotive engine is attached by a free joint to a point in one of the radii of the driving wheel; shew that the pressure upon the axis of the wheel tending to force it forward is the same for symmetrical positions of the piston rod as the wheel revolves, whether the rod be pushing or pulling.

(56). A hemisphere of given radius rests upon the top of a rough fixed sphere; find the radius of this sphere such that the equilibrium may be neutral.

(57). If a cylinder has its base united concentrically to the base of a hemisphere of equal radius, find the height of the cylinder in order that the solid may stand upon a smooth horizontal plane on any point of its spherical surface.

(58). A sphere of radius r rests upon the concave surface of a sphere of radius R; if the first sphere be loaded so that the height of its centre of gravity from the point of contact be $\frac{3}{2}r$, find R such that the equilibrium may be neutral.

(59). Out of a solid right cone is cut another cone having coincident base and axis; the remaining solid is placed like an inverted wine-glass upon a pointed rod: find the relation between the lengths of the axis of the cones, according as the equilibrium is stable, neutral, or unstable.

(60). When the solid cut out of a cylinder by two planes passing through the axis is in equilibrium with its cylindrical surface placed upon an horizontal plane and edge uppermost, prove that the equilibrium is stable however small be the inclination of the planes.

DYNAMICS.

SECTION I.

PRELIMINARY.

(1). It has been defined (Art. 3, *Statics*) that when the constituent particles of a body occupy different positions in space at successive instants of time, the body is in motion. Its particles are said to be animated with *velocity*.

If during its motion a material particle always describes any given space in the same time, wherever in its course the space be taken it is said to be moving uniformly, or to have a *uniform velocity; i.e.* its velocity at every instant is the same throughout the motion.

On the contrary, if a given space, wherever taken, be not passed over in the same time, the motion is not uniform; or the velocity of the particle varies from instant to instant.

Supposing the motion to be such that the given space is passed over more quickly in the latter part of the particle's course than in the first, the particle's velocity would be said to have increased, and *vice versâ*.

It is not difficult, from such considerations as the preceding, to form a distinct idea of velocity as being a property of a moving particle, differing in degree from instant to instant; its magnitude at any one moment being perfectly independent of the subsequent movement of the particle.

(2). Def. The velocity of a particle in motion is, at a proposed instant, proportional to the space through which it would carry the particle in any given interval of time, provided it remained uniform throughout that interval, and of the same magnitude as at the proposed instant.

In accordance with this definition, we could correctly take as the *measure* of velocity the space through which it would, if continued uniform, carry a particle in *any given duration* of time. It is generally found convenient to take the unit of time which is in use for such interval: by the unit of time is meant that particular interval in terms of which all other durations of time are measured. Thus under the common convention, where one second is the unit of time, if we observed a ball in motion at a particular instant, and had the means of knowing that if it continued to move for the next second exactly as fast as at the instant under consideration it would in that interval pass over 2 feet, we should measure its velocity at the *proposed instant* by the length 2 feet; and this we should do, whether the ball's real motion were uniform or not during the second.

In thus measuring velocity we do not deviate from our ordinary modes of thinking and speaking: when a railway train is said to be going at the rate of 30 miles an hour, it is not meant to convey the idea that the train really passes over 30 miles in the next hour, but merely that it would do so provided that it moved for that time with the speed it has at the moment alluded to; the fact of its being obliged to stop at a station before the lapse of an hour does not alter our notion of its speed at that time.

3. The symbol usually adopted to represent the measure of velocity, is v; and when we meet with the expression $v = b$ in relation to a moving particle, we should understand that the particle has such a velocity at a proposed instant as would carry it through b *units of space* in the next *unit of time*, provided that it moved through that unit of time with a velocity equal to that which it has at the proposed instant.

Thus velocity is measured by a length; it has also direction, the direction in which the particle is moving under its influence; it can therefore, like forces in Statics, be represented by a straight line.

4. It is the object of Dynamics to investigate the laws from which we may deduce the effects of a system of forces, as regards the movement of the particle or system of particles upon which they act.

The first step in this inquiry is provided by the assertion which is generally known under the name of the

First Law of Motion,

A material particle if at rest will remain at rest, and if in motion will continue to move uniformly in the same straight line, unless it be acted upon by some external force.

From our definitions already given of Force and Velocity, we acknowledge the presence of the first whenever we observe a change of any kind in the second; they are therefore two objects whose existence we can readily detect and recognize: the relation which the one bears to the other must of course depend upon the laws which govern the material world; we can only discover those laws or their results by a careful observation of natural phenomena.

The Law of Motion just enunciated is merely a convenient form of asserting one of the conclusions to which observers of nature have been led respecting the connexion between forces and matter. It is simple but important; it states that no portion of matter can move itself from rest, or in any way alter the nature of its own motion when it is in motion. All change that may be observed in its state either of rest or of motion is to be referred to some force from without. This inability of matter to alter in any way its own state of rest or motion is commonly known by the term *Inertia.* If then a single force be applied to a particle of matter, a change in the particle's velocity must ensue and be due to this force alone: *à priori,* as far as we know, the relation between the

force and the change it produces might be any whatever although definite. It cannot be made the subject of abstract definition.

5. The relation which is really found to exist between forces and their effects upon velocity is asserted in the following words:

Force when uniform is proportional to the velocity which it either generates or destroys in a given time in the particle to which it is applied.

It must here be understood, that a force acting upon a material particle during any time is said to be *uniform* or *constant*, when in any equal intervals whatever it generates or destroys in the particle equal quantities of velocity: and that one force is said to be dynamically equal to another, when they both in a given time generate or destroy the same velocity in the same particle.

6. This assertion shews us that if we chose to consider forces simply with reference to the effect which they produce in altering the velocity of a material particle, we could very correctly *measure* them by the velocity which they respectively generate or destroy in that particle in a unit of time: and this is in fact the convention which is usually made; but then force so measured receives the distinctive appellation of *accelerating force*.

7. What is really thus measured ought more properly to be termed the *accelerating effect* of the force upon the particle: a measure of this is by no means a measure of the force itself; for it is found by experiment that a given force when applied to a number of heavy particles in succession generally produces very different changes of velocity in the different particles; this is due to a difference in the particles themselves, whereby they are said to differ in *mass*, a term which will hereafter be made the subject of definition and measurement. Thus it appears that the same force, estimated as an *accelerating force*, will have different values according to the difference in mass of the

particles upon which it acts. At present, when speaking of a force in reference to its action upon a material particle, we shall suppose its accelerating effect with respect to that particle to be known: it will belong to a future section to explain how, when a force is given, the measure of its accelerating force may be obtained for any given particle whatever.

8. The preceding assertion of the proportion between force and the velocity generated or destroyed by it might with propriety be termed a law of motion: but it is found more convenient to consider it as a deduction from a more comprehensive law, called the *Second Law of Motion,* which declares that,

When any number of forces act simultaneously upon a material particle in motion, the instantaneous quality of each force as regards its intensity and direction remains the same as if it had acted alone upon the particle at rest.

Let f_1, f_2, f_n, represent n forces, such as would generate during a very small interval of time τ velocities v_1, v_2,. . . .v_n, respectively, in a particle of matter, whether moving or not; suppose them to act simultaneously upon the particle and in the same direction for the time τ, then each force f will, during that time, according to the law just enunciated, generate a velocity v in the particle, the same as it would have done had it acted alone for that time, and therefore the velocity generated by all the forces acting together will be the sums of the velocities due to them separately; that is, a force $f_1 + f_2 + \ldots + f_n$ will generate in time τ a velocity $v_1 + v_2 + \ldots + v_n$. If we put $f_1 = f_2 = \ldots . = f_n$, we shall have $v_1 = v_2 = \ldots . = v_n$ by Art. (5), and the preceding result takes this form: if a force f_1 will generate in a particle a velocity v_1 in time τ, then a force nf_1 will generate a velocity nv_1 in the same; which proves the assertion already made of the proportionality between force and the velocity it generates in a particle in a given time.

(9). We have seen that all forces are known by their producing or tending to produce *change* in a body's velocity: in general they require to act for a finite time in order to produce an appreciable change, and are therefore termed finite forces; but there is a class of forces, called *Impulsive* forces, which produce a finite change in an indefinitely small time: for instance, when a ball is struck by a bat it is acted upon by an impulsive force during the instant only that the ball and bat are in contact, but the change produced during that time in the magnitude and direction of its velocity is very apparent.

Impulsive forces differ not at all from Finite forces in character; they are only peculiar in the short duration of their action and in their intensity, which enables them to produce a finite effect in that time; they are, equally with finite forces, the subject of the laws which have been enunciated in Art. (8). Therefore as the time of action of all impulsive forces is to our senses the same, in every case too small to admit of our measuring it, we shall be right in considering them proportional to the velocity which they generate in a particle in that time. As then we measure the *accelerating effect* of a *finite force* by the velocity which it generates or destroys in a particle in a unit of time (Art. 6), we measure the *accelerating effect* of an *impulsive force* by the velocity it generates or destroys in the time of its action, an instant.

It is manifest that these two ways of measuring the two classes of forces, although based upon the same principle, render it impossible to compare individual forces of the one class with those of the other: but it is also equally manifest that such a comparison can never be needed in practice, for if impulsive and finite forces be applied together to a particle, the effects of the first would be produced before those of the latter were in the smallest degree appreciable; the two systems must therefore be considered as applied in succession and not together.

It is almost superfluous to remark that forces, whose accelerating effects upon the same particle are equal, must be statically equal.

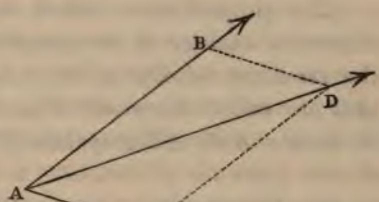
(10). We now come to the consideration of the important problem which our science proposes to us:

Given the position of a particle in space at any time, and the magnitude and direction of the velocity which it has at that instant, also the forces which are then acting upon it; find the motion of the particle during the next very small interval of time.

If we can solve this, we can of course repeat the operation at the end of this interval, and again at the end of a second such interval and so on, and by this means pursue the particle through the whole of its course. The use of the Differential and Integral Calculus renders this proceeding tolerably easy in most cases. At present however we shall confine our attention to cases, wherein the particle moves in one plane and where the solution of the problem can be accomplished by simple processes.

The remainder of this section is occupied by the demonstration of certain principles upon which the solution of this problem in all cases depends.

11. *If two impulsive forces act simultaneously upon a particle* A *at rest, and be such as would separately generate in* A *velocities represented in magnitude and direction by* AB, AC *respectively, they will together generate in* A *a velocity represented in magnitude and direction by* AD *the diagonal of the parallelogram described upon* AB *and* AC.



For since AB and AC represent the velocities generated by the two forces respectively, they likewise represent the forces themselves (Art. 9); therefore, by the parallelogram of forces, AD represents the *Statical* and consequently, by the Second Law of Motion, the *Dynamical* resultant of the forces; but

this is measured by the velocity it produces. Hence AD represents in magnitude and direction the velocity which the two forces acting together generate in the particle.

This proposition may be stated somewhat differently thus:

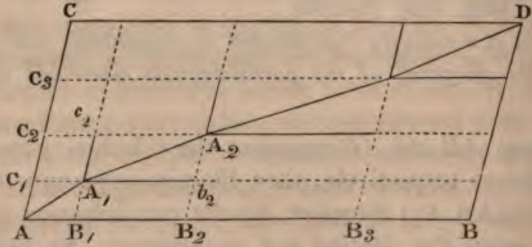
If any two causes, such as would separately make the particle A *begin to move in directions* AB *and* AC *with velocities* AB *and* AC *respectively, coexist, they will then make the particle move in the direction* AD *with the velocity* AD.

One of these causes, as that which tends to produce the velocity AB, may manifestly be simply the tendency of the body to move along AB with a velocity AB, in consequence of a previous motion in that direction (First Law of Motion), but then the other cause must be either a single impulsive force, or the resultant of a set of impulsive forces.

Conversely, *if a particle* A *be moving at a certain instant in a direction* AD *with a velocity* AD, *we may suppose the cause which produces this motion to be equivalent to the coexistence at that moment of two causes such as would separately make the particle begin to move in directions* AB *and* AC *with velocities* AB *and* AC *respectively.*

This proposition is called the *Parallelogram of Velocities.* From the analogy of the parallelogram of forces AD is called the *resultant velocity* of the velocities AB and AC; and AB, AC are called the *resolved parts* of the velocity AD in the directions AB, AC respectively.

12. *If two causes act simultaneously throughout an interval*



of time t *upon a particle which at the beginning of the time* t *was*

at rest at A, *they being such as would if they acted separately cause the particle to move from* A *to* B *and from* A *to* C *respectively in the same duration of time* t, *then will they together cause the particle to be at the end of* t, *at the point* D *which is the extremity of the diagonal of the parallelogram formed upon* AB *and* AC.

The cause which would alone make A move from A to B in time t, might be made up of an initial velocity in the direction AB, impulsive forces acting at certain instants in the same direction and finite forces acting for the whole time or parts of the time; or it might consist of only one or two of these parts: however in either case its effect would be that A would under its action move from A to B with a velocity varying from instant to instant.

Let the time t be divided into n intervals each equal τ so that $t = n\tau$, and let AB_1, B_1B_2, B_2B_3 ...... be the spaces through which the particle A would move in the first, second, third, &c. interval τ respectively, under the influence of this cause acting alone.

Now suppose another cause, consisting of a series of impulsive forces acting upon the particle A at the beginning of each interval τ, to make A move from A to B_1 uniformly in the first interval τ, and from B_1 to B_2 in the second interval τ and so on; it is quite clear that if τ be small compared with t, that is if n be large, the motion of A under the action of this second cause will be nearly that produced by the first; and the greater n be, and therefore the smaller τ, the more nearly will the second motion approach to the nature of the first; and if τ be made very small by increasing n sufficiently, we may suppose the effect of the second cause upon the movement of A to coincide with that of the proposed one: but this second cause consists of a succession of impulsive forces, capable of making A move in direction AB, with velocities which are in the first, second, ... small interval τ, proportional to AB_1, B_1B_2....

Similarly, if AC_1, C_1C_2, C_2C_3, ... be the spaces through which A would move in first, second, third, ... interval τ respectively under the action of the cause which would alone

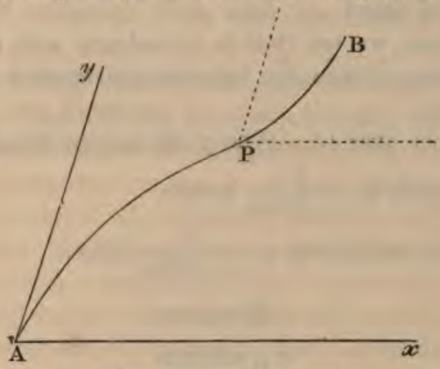
make it move from A to C in time t, we may, when τ is made small enough as before, substitute for this cause a succession of impulsive forces acting at the beginning of each interval π and making A move uniformly through those small intervals with velocities proportional to AC_1, C_1C_2, ...

Suppose, again, these two sets of impulsive forces to act together upon A, at the beginning of each of the intervals τ. Let AA_1 be diagonal of the parallelogram described upon AB_1, AC_1; draw A_1b_2, A_1c_2 respectively parallel and equal to B_1B_2, C_1C_2, and let A_1A_2 be the diagonal of the parallelogram described upon A_1b_2 A_1c_2: let also, in a similar way, parallelograms be constructed for each pair of lengths B_2B_3 and C_2C_3, B_3B_4 and C_3C_4, &c., it is sufficiently evident that A_n will coincide with D.

Since at beginning of the first interval τ A is simultaneously acted upon by two causes which would separately produce in it velocities proportional both in magnitude and direction to AB_1, AC_1 respectively, therefore by the parallelogram of velocities the velocity that is really communicated to it is proportional both in magnitude and direction to AA_1, and as by supposition the resolved velocity along AB_1 carry it from A to B_1 in time τ, so the resultant velocity carries it from A to A_1 in that time. At the beginning of the second interval τ, then, the particle is at A_1; and two causes now act simultaneously such as would separately make the particle move uniformly from A_1 to b_2 and from A_1 to c_2 respectively in the next interval τ; hence as before its real motion in that time will be from A_1 to A_2. By proceeding with this reasoning we can prove that under the simultaneous action of the supposed two systems of impulsive forces, the particle must be at A_3, A_4 .... A_n at the end of the third, fourth, nth intervals τ respectively: this result is quite independent of the length of τ; but we have seen that if τ be made indefinitely small these two systems of simultaneous impulsive forces must coincide in their effects with the two proposed causes acting simultaneously upon the particle. Therefore the simultaneous action of the two proposed causes upon the particle during the time t will cause it to be at D at the end of that time.

13. Suppose APB to be the course which a particle describes under the action of certain forces, P any point in that course, t the time of the particle being at that point.

The velocity which the particle has at the time t may be resolved into two parts parallel to any two given directions Ax



and Ay respectively; the forces acting upon it at the same time may also be resolved into two parallel to the same directions; hence at time t the particle may be conceived to be at P under the simultaneous action of two causes which would separately make it move parallel to Ax and Ay respectively. The same may be said of it at any other point and time. Therefore the whole motion may be conceived to be produced by the continued simultaneous action of two causes which would separately make the particle move in the given directions.

If then these resolved forces be known for every instant, and the united effects of them and the initial resolved velocities of the particle can be calculated for a given time t in their separate directions Ax and Ay, the actual position of the particle under their united action will be known by the last proposition.

This is the method generally pursued in Dynamics for investigating the motion of a particle; it depends for its efficacy in a great measure, as has before been said, upon the application of the Differential and Integral Calculus. In the following pages our attention will be confined to cases of the simplest kind.

Examples to Section I.

(1). A man rides, at a uniform pace, 15 miles in 2 hours; if his velocity be 5, and a yard be the unit of length, what is the unit of time?

In this case, we see that in accordance with the definition of the measure of velocity, the man traverses 5 yards in the unit of time.

But he goes over 15×1760 yards in two hours, and therefore over 5 yards in $\dfrac{2}{3 \times 1760}$ hours.

Hence the unit of time is $\dfrac{2}{3 \times 1760}$ hours

$$= \tfrac{20}{880} \text{ minutes}$$

$$= \tfrac{1}{44} \text{ minutes}$$

$$= \tfrac{15}{11} \text{ seconds}$$

$$= 1.36 \text{ seconds.}$$

(2). A person travelling on a railway with a velocity v, wishes to fire at an object in a particular direction; supposing the course of the bullet to be rectilinear, and its velocity uniform, equal v' in the direction of the barrel of the gun, determine the direction in which he should aim.

The bullet will move under the action of two causes; the first of which, if it acted alone, would carry it along uniformly with velocity v in the direction in which the carriage is moving; the other would take it uniformly with a velocity v' in the direction of the gun-barrel; its real direction then will be, by the parallelogram of velocities, in the direction of the diagonal of the parallelogram, whose sides are in the first two directions and are proportional to v and v' respectively.

To secure that this diagonal shall pass through the object required, we may make the following construction:

Supposing A to be the position of the person who fires

the gun, and P the object; take AB in the direction of the carriage's motion, proportional to v, and describe a circle about B with a radius proportional to v': let D be the point where this circle meets AP; complete the parallelogram $ABDC$. If the gun be fired in the direction AC, the course of the ball will evidently, from what has been said, be the line AP.

Since the circle will generally meet AP in two points, there will be two directions in which the gun may be fired in order to hit P; subject to exceptions which can be easily interpreted.

SECTION II.

FORMULÆ FOR RECTILINEAR MOTION.

14. We use the following symbols to represent the quantities which have been already the subject of our definitions. When we enunciate the formulæ expressive of the relations between these quantities, t generally denotes the time of an event's happening, a particular instant: it is the *number* of units of time, whatever that unit may be, which have elapsed at the proposed instant, since some given fixed point of time. When the contrary is not particularly stated, the interval called a second is taken as the unit of time. v represents the *measure* of the velocity of any particle at a proposed instant: as such, its full meaning has been already given, but it may be as well here to repeat that it is the *number* of units of length through which it would carry the particle in a unit of time, provided it remained constant for that time. Unless the contrary be particularly stated, the length called a foot is taken as the unit of length. Thus if a particle be said to have a velocity v at a particular moment, it is meant that it would, with the velocity which it has at that moment continued constant, pass over v feet in one second.

f is generally employed to indicate the *measure* of an accelerating force; it is therefore (Art. 6) *the velocity* which the force acting upon a single particle will generate in it in a unit of time; and as the measure of velocity is a length, therefore f, the measure of accelerating force, is a length: thus, using the conventional units of time and space, *i.e.* seconds and feet, an accelerating force, whose measure is f acting for one second uniformly upon a particle which was at first at rest, will generate in it by the end of that time a velocity f, or, more explicitly, a velocity which would, if continued uniform, alone carry the particle over f feet in a second of time. Of

course in generating this velocity the force has moved the particle through some space; we shall see by-and-bye that this space in all cases $= \frac{1}{2}f$ feet: this fact may be usefully remembered.

s is commonly taken to represent the distance which a particle may have traversed in any assigned time, or under any proposed circumstances.

Although these symbols generally stand for the quantities which have just been attributed to them, any others may of course be substituted for them at pleasure; it is only here meant that they constitute the symbols of our ordinary notation. Besides these, particular letters have been conveniently used to designate particular forces. For instance the letter g stands for the *accelerating force* of *gravity*, which is found to be always the same at the earth's surface upon every particle; its numerical value found by experiment is 32.2 when feet and seconds are the units of length and time respectively. This means, in accordance with the explanation just given of the letter f when it stands for accelerating force, that if gravity be allowed to act upon a particle for one second, it will generate or destroy in that particle a velocity g, or a velocity which would alone carry the particle in one second over g (= 32.2) feet.

15. All these symbols are numerical, and represent the number of times which the unit, in terms of which they are estimated, enters the quantity for which they stand; they will therefore change their values when these units are for any reason changed, although the quantities denoted by them remain the same: thus any particular velocity being considered, the v which stands for it may change its numerical value for two reasons.

1st. The unit of time may be changed; for instance it may be convenient to take 1 minute as the unit in which to measure time instead of 1 second, as is ordinarily the case: now the v stands for the distance through which the velocity continued uniform would carry a particle in a unit of time; it would

manifestly carry it 60 times as far in 1 minute as in 1 second; hence in the first case the numerical value of v would be 60 times that in the second, although the same velocity is represented in both cases.

2nd. The unit of distance might be altered; a yard might be employed instead of a foot: if the unit of time remained the same, as 1 second, and therefore the distance denoting the velocity the same, still its numerical value would become $\frac{1}{3}$ of what it was, because distance would be measured in terms of yards instead of feet.

If both these changes in the units were made at once, a change would generally take place in the value of v; but if the one unit were always increased in the same proportion as the other, the value of v would manifestly remain unaltered.

Again, in regard to a particular force, a change in the unit of time will make a more complicated change in its value: we have seen that it is measured by the velocity it generates in a particle in a unit of time; if then this unit be increased, manifestly the absolute velocity generated in that time is increased in the same proportion: moreover, by this increase of the unit the measure of any given velocity is increased, also in the same proportion, because by definition this measure is the distance through which the given velocity will carry the particle in the unit, and which must be greater for the same velocity, the greater the magnitude of the unit: hence, had the velocity which stands for the force remained the same after, as before, the change in the unit of time, the numerical value of the force would still be changed in direct proportion to the length of the unit; but as this velocity is itself changed in the same proportion, the numerical value of the force must be changed in proportion to the square of the unit.

To take a familiar example:

The value of g which stands for the force of gravity is the numerical value of the velocity which it will generate in a particle in one second = 32.2, because the velocity so generated will alone carry the particle over a space 32.2 feet in one second.

Now when 1 minute is taken as the unit of time, although the velocity generated by gravity in 1 second is necessarily the same as it was when 1 second was the unit, still its numerical value is 60 times as great, because it is measured by the distance it will carry a particle in the unit; therefore the velocity generated in 1 second is 60×32.2 feet, and that generated in 1 minute must be $60 \{60 \times 32.2\}$; hence the value of g is in this case $60^2 \times 32.2$ instead of 32.2.

In general, if f_1 and f_2 represent the numerical values of the same force in two cases where the units of time are as m_1 to m_2 respectively, similar reasoning to the above would shew that

$$f_2 = \left(\frac{m_2}{m_1}\right)^2 \times f_1.$$

If also the units of length in the two cases were as n_1 to n_2, we should get

$$f_2 = \frac{n_1}{n_2}\left(\frac{m_2}{m_1}\right)^2 \times f_1.$$

16. *If a particle not acted upon by any force be moving with a velocity* v, *and* s *be the space it describes in* t *units, then*

$$s = vt.$$

For v represents the space through which the velocity would alone carry the particle in one unit of time, if it remained of the same intensity throughout that unit: in this case, as no force is acting, v does remain of the same intensity throughout the motion (First Law of Motion); therefore it carries the particle through a space v in every unit and though a space $t \times v = vt$ in t units.

17. *If a force act uniformly during* t *units of time upon a particle which was initially at rest, and if* v *be the velocity generated in the particle at the end of that time, and if* f *be the measure of the accelerating effect of the force upon the particle, then*

$$v = ft.$$

For f is by definition the velocity which the force will generate in any unit of time; therefore in t units it will generate a velocity ft.

18. *If* s *be the space through which the particle originally at rest is carried by this accelerating force* f, *acting uniformly upon it for* t *units, then*

$$s = \tfrac{1}{2}ft^2.$$

To prove this, let the time t to be divided into n equal intervals, each equal τ; then by the last article the velocity which the particle has, in its course, acquired at the end of the 1st, 2nd, 3rd,. . . . rth interval respectively, is

$$f\tau,\ \ f \times 2\tau,\ \ f \times 3\tau, \ldots .\ f \times r\tau.$$

Suppose now that the particle, instead of moving as it really does under the action of the force, moved during each interval with the velocity which it has in the real motion at the beginning of that interval: it is clear that in this supposed motion the particle would always, excepting at the beginning of each interval, be moving slower than it does in the real motion, and therefore the distance described by it would be less. But if s_1 be the whole distance described in this supposed case, it equals the sum of the spaces described in the successive intervals τ, the velocity during the first interval being 0, during the second $f\tau$, during the third $2f\tau$, and so on; therefore

$$\begin{aligned} s_1 &= f\tau \times \tau + 2f\tau \times \tau + \ldots\ldots + (n-1)f\tau \times \tau \\ &= f\tau^2 \left\{1 + 2 + \ldots\ldots + (n-1)\right\} \\ &= \frac{n(n-1)}{2} f\tau^2 \\ &= \frac{\left(1 - \dfrac{1}{n}\right)}{2} fn^2\tau^2 \\ &= \left(1 - \frac{1}{n}\right)\frac{ft^2}{2} \qquad \text{because } t = n\tau \\ &= \frac{ft^2}{2} - \frac{1}{n}\frac{ft^2}{2}. \end{aligned}$$

Similarly, if we suppose the particle to move during each interval with a velocity which, in the real motion, it has at

the end of that interval, it will always, excepting at the ends of the intervals, be moving faster than it does in the real motion, and if s_2 be the space so described, it will be greater than the space required to be found.

But the velocities at the end of 1st, 2nd, ... intervals are $f\tau$, $2f\tau$, &c.; and therefore

$$\begin{aligned} s_2 &= f\tau \times \tau + 2f\tau \times \tau \ldots\ldots n f\tau \times \tau \\ &= f\tau^2 (1 + 2 + \ldots\ldots + n) \\ &= \frac{n(n+1)}{2} f\tau^2 \\ &= \frac{1 + \frac{1}{n}}{2} f n^2 \tau^2 \\ &= \left(1 + \frac{1}{n}\right) \frac{ft^2}{2} = \frac{ft^2}{2} + \frac{1}{n} \frac{ft^2}{2}. \end{aligned}$$

Now by what has been said, s the space which the particle really does describe in the time t must be intermediate between s_1 and s_2, whatever be the lengths of the intervals τ, and therefore it is intermediate to them when τ is indefinitely small, *i.e.* when n is indefinitely large: but as n is increased it is seen that s_1 and s_2 both approach the value $\frac{1}{2}ft^2$, and when n becomes indefinitely large they both become indefinitely near to that value, therefore s which is always between them must be that value itself.

19. If σ represent the space through which a force f would carry a particle from rest in a unit of time, its value will be obtained from the preceding formula by putting $t = 1$; this substitution gives

$$\sigma = \tfrac{1}{2}f,$$

a result which has been already alluded to Art. (14).

20. Since $$s = \tfrac{1}{2}ft^2,$$
and $$v = ft,$$

are simultaneous relations between s, f, v, and t, we obtain by the elimination of t a third relation

$$v^2 = 2fs.$$

21. In the preceding formulæ we have supposed f to act upon a particle which was initially at rest. Had it been initially in movement with a velocity V, then if f acted upon it for t units of time in the direction of its motion, it would generate an additional velocity ft (Second Law of Motion), or if it acted in the contrary direction it would destroy a velocity ft; and therefore at the end of t units the velocity of the particle would be $V \pm ft$, where the + or − sign must be used according as the force acts in the same direction as the particle is moving, or the contrary.

22. In order to find the space described by this particle, which starts with a velocity V and is acted upon for t units of time by a force f, we can employ the method of Art. (18). The real velocity of the particle at the end of the 1st, 2nd,...rth interval τ respectively would be $V \pm f\tau$, $V \pm 2f\tau$, ... $V \pm rf\tau$, where the + or − sign must be taken according as the force acts in the same direction as the particle is initially moving, or the contrary. We should thus have

$$\begin{aligned} s_1 &= V \times \tau + (V \pm f\tau) \times \tau + \ldots\ldots + \{V \pm (n-1) f\tau\}\,\tau \\ &= Vn\tau \pm f\tau^2 \{1 + 2 + \ldots\ldots + (n-1)\} \\ &= Vt \pm \left(1 - \frac{1}{n}\right)\frac{ft^2}{2}. \end{aligned}$$

Similarly, $$s_2 = Vt \pm \left(1 + \frac{1}{n}\right)\frac{ft^2}{2}.$$

Hence, by the same reasoning as in the simple case, if s represent the actual space described by the particle in the time t,

$$s = Vt \pm \tfrac{1}{2}ft^2,$$

the + or the − sign being used according to the foregoing explanation.

23. From these two general relations between v, s, f, t, we easily get, by eliminating t,

$$v^2 = V^2 \pm 2fs,$$

where we are guided to the sign to be taken by the same rule as before.

24. If a particle starting in a given direction with a velocity V be acted upon for t units of time by a force f in a direction exactly opposite, its velocity v at the end of that time is

$$v = V - ft,$$

therefore as t increases, v diminishes, or the longer f acts the smaller the velocity becomes; in fact the force is employed in gradually destroying the velocity, and at the end of an interval $t = \dfrac{V}{f}$, a numerical quantity, the formula gives us $v = 0$; the velocity of the particle is then all destroyed. If the force still continues to act, after having thus brought the particle to rest, it will make it move henceforward in its own direction, which is exactly opposite to that of the particle's first motion: but if we make t in our formula greater than $\dfrac{V}{f}$, we get

$$v = -(ft - V) = -f\left(t - \frac{V}{f}\right),$$

a negative quantity, whose numerical value is evidently that of the velocity which the force f would generate in the interval $t - \dfrac{V}{f}$, that is, in the interval which has elapsed since the particle was brought to rest, and therefore of the velocity which the force has at time t generated in the particle in the backward direction. Hence, if we agree that the signs + and −, when prefixed to symbols of velocity, shall be distinctive of opposite directions, our formula will apply to all circumstances of a particle in rectilinear motion. This is the convention usually made. It is also extended to the symbols of

space, force, and time, where its adoption is justified by similar reasoning to that which we have just gone through.

It may not be useless to give here an instance of the consistency of this convention with regard to time.

Suppose a particle to be at A, moving with a velocity V towards K, and suppose it to be acted upon by a constant force f in the same direction during t units of time; then, if P_1 be its position at the end of that time, we have by our formula,

$$AP_1 = Vt + \tfrac{1}{2}ft^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Supposing the movement of the particle at A, and afterwards, to be merely the continuation of a previous motion, then if P_2 were the position of the particle t units of time before it came to A, it ought, according to our convention, to be given to us by putting $-t$ instead of t into our formula; therefore

$$\begin{aligned} AP_2 &= -Vt + \tfrac{1}{2}ft^2 \\ &= -(Vt - \tfrac{1}{2}ft^2) \quad\ldots\ldots\ldots\ldots (2). \end{aligned}$$

Hence we learn that the numerical value of the length AP_2, without regard to its negative sign, is $Vt - \tfrac{1}{2}ft^2$.

Now if V' were the velocity of the particle when it was at P_2, since by this and the force f acting upon it, it must by supposition get to A in t units of time, we have by formula,

$$P_2A = V't + \tfrac{1}{2}ft^2.$$

Also V' must be such that, when after t units of time the particle comes by its action and that of f to A, it shall equal V, i.e.

$$V' + ft = V;$$

therefore $$V't = Vt - ft^2,$$

and by substitution $$P_2A = Vt - \tfrac{1}{2}ft^2,$$

which exactly agrees with the numerical value of AP, given by the substitution of $-t$ for t in the original formula. The negative sign prefixed to this value in (2) shews, according

to the convention, that AP_2 must be taken in a direction opposite to that of AP_1. Hence it is seen that our convention, with regard to the interpretation of the signs + and −, renders the formulæ very general and in no degree inconsistent.

25. It may be useful to exhibit together the formulæ which have been proved in this section. They are

$$\left.\begin{aligned} s &= vt \\ v &= ft \\ s &= \tfrac{1}{2}ft^2 \\ v^2 &= 2fs \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(A)},$$

$$\left.\begin{aligned} v &= V \pm ft \\ s &= Vt \pm \tfrac{1}{2}ft^2 \\ v^2 &= V^2 \pm 2fs \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(B)}.$$

Examples to Section II.

(1). If the numerical value of an accelerating force be f, when one second is taken as the unit of time, what is its numerical value when half-a-second is taken as the unit?

An accelerating force is measured by the velocity which it will generate in a particle in a unit of time, while the velocity itself is, in consequence of our conventional measurements, measured by a distance which is equal to twice that through which the force would move the particle from rest in the same time (Art. 19).

Hence, if f' be the new measure of the force,

$$f : f' :: 2 \text{ (space described in } 1'') : 2 \text{ (space described in } \tfrac{1}{2}'').$$

Now spaces described from rest, under the action of a given force, vary as the squares of the times of describing them; therefore $$f : f' :: 1 : \tfrac{1}{4},$$
or $$f' = \tfrac{1}{4}f.$$

(2). A particle descends from rest under the action of gravity, and describes in the nth second of its fall a space equal to p times the space described in the $(n-1)$th second; find the whole space fallen through.

Generally we have, if s be the space in feet fallen through in t seconds from rest,

$$s = \tfrac{1}{2}gt^2.$$

Hence the space fallen through in n seconds will be, σ suppose,

$$\sigma = \tfrac{1}{2}gn^2;$$

therefore, space in the nth second $= \tfrac{1}{2}g\,\{n^2 - (n-1)^2\}$

$$= \tfrac{1}{2}g\,(2n-1):$$

similarly, space in the $(n-1)$th second $= \tfrac{1}{2}g\,(2n-3)$;
therefore, by the question,

$$\tfrac{1}{2}g\,(2n-1) = p \times \tfrac{1}{2}g\,(2n-3);$$

therefore $$n = \frac{3p-1}{2\,(p-1)},$$

and $$\sigma = g\,\frac{(3p-1)^2}{8\,(p-1)^2}.$$

(3). A person ascending in a balloon lets fall a stone when at a given height. Determine the time of the stone's reaching the ground supposing the velocity of the balloon at the given altitude known. Explain the meaning of the negative value of t.

Let h be the height of the balloon when the stone is let fall, v its ascending velocity, which is consequently communicated to the stone.

If s be the distance of the stone *above* this height h, t'' after being freed from the balloon, our formula gives us

$$s = vt - \tfrac{1}{2}gt^2.$$

If in this equation we put $-h$ for s, we shall obtain the value of t for the instant of the stone's reaching the ground, which is at a distance h *below* the point from which the stone

started. By a little transformation this gives us

$$t^2 - \frac{2v}{g}t = \frac{2h}{g};$$

therefore $$\left(t - \frac{v}{g}\right)^2 = \frac{v^2 + 2gh}{g^2};$$

and therefore $$t = \frac{\sqrt{(v^2 + 2gh)} + v}{g},$$

or $$= -\frac{\sqrt{(v^2 + 2gh)} - v}{g}.$$

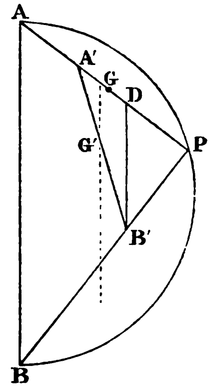
It is quite evident that if the stone, instead of rising to the height h in the balloon, and of then being dropped from it, had been projected upwards from the surface of the earth with a velocity $= \sqrt{(v^2 + 2gh)}$, its condition at the height h and its subsequent movement would have been the same as in the proposed case. But the time it would take in rising so high would be $\frac{\sqrt{(v^2 + 2gh)} - v}{g}$, which is exactly the same in *numerical* value as the negative value of t above given.

Now the analysis by which we have solved the given problem did not contemplate any discontinuity in the stone's movement, it therefore gave us two times at which the stone was at the earth's surface, the one when it started and the other when it came to it again; the first of these is affected with a negative sign, shewing that it occurred *before* the time of arriving at the height from which we dated our time.

(4) Two equal bodies begin at the same instant to descend from rest along chords AP, PB of a semicircle, the diameter AB of which is vertical. Shew that their centre of gravity will descend vertically.

Bisect AP in G, then, since the bodies are equal, G was the position of their centre of gravity at the beginning of the motion. Let $A'B'$ be any contemporaneous positions of the bodies, and let the vertical through G meet $A'B'$ in G'; it is sufficient for the proof of our problem to shew that G' bisects $A'B'$.

The accelerating forces upon A' and B' are the resolved



parts of gravity in the directions AP and PB respectively, they are therefore proportional to AP and PB, since AB is vertical. Also we see by the formula $s = \frac{1}{2}ft^2$, that the spaces described in the *same* time by bodies under the action of different accelerating forces are proportional to those forces; therefore we have, drawing $B'D$ parallel to AB,

$$AA' : PB' :: AP : PB :: PD : PB' \text{ by similar triangles,}$$

therefore $$AA' = PD,$$

and G bisects $A'D$, and therefore G' bisects $A'B'$. Q.E.D.

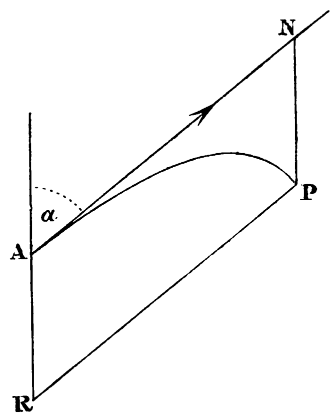
SECTION III.

CURVILINEAR MOTION.

26. A HEAVY particle projected into the atmosphere from any point on the earth's surface describes a path whose nature it is very difficult accurately to determine. The difficulty arises from the presence of the retarding force of the air's resistance; if we omit this force and consider the particle to be moving in a vacuum under the action of the earth's attraction only, we obtain a very simple case of Curvilinear Motion, which is generally denominated the

Motion of Projectiles.

27. To investigate the path of a projectile, we will suppose A to be the point from which the particle is projected; V the velocity and AT the direction of its projection; let AT make an angle a with the vertical AR.



Since we neglect the resistance of the air, the only force acting upon the particle will be that of gravity $= g$ in a vertical direction: hence, the motion of the particle must evidently be in the vertical plane passing through AT the direction of projection, for there is no force to draw it out of that plane.

If the particle were unaffected by the action of gravity, it would by the First Law of Motion move uniformly along

the line AT with the velocity V which it had initially at A: let N be the point in the line where it would thus arrive in t'' after starting from A: again, were it affected by gravity, but possessed of no initial velocity in any direction, it would fall in a vertical direction AR; let R be the point where it would under these circumstances arrive in t'': complete the parallelogram $RANP$; then, in the actual motion, since both these causes act together, by the Second Law of Motion, or rather by Art. (12) which contains a deduction from it, P is the true position of the particle at the end of that time t.

Now $\quad AN = Vt, \ldots . \{\text{Art. (16)}\},$

and $\quad AR = \frac{1}{2}gt^2,$

therefore $\quad \dfrac{AN^2}{AR} = \dfrac{2V^2}{g};$

or $\quad PR^2 = 4\dfrac{V^2}{2g}.AR \ldots\ldots\ldots\ldots\ldots\ldots (1).$

But AR is the abscissa and PR the ordinate corresponding to the position of the particle at any time t, and the relation (1) which exists between them is exactly that which exists between the abscissa and ordinate of any point in a parabola of which AR is a diameter.

Hence the particle in its movement describes a parabola whose axis is vertical and the distance of whose focus from $A = \dfrac{V^2}{2g}$.

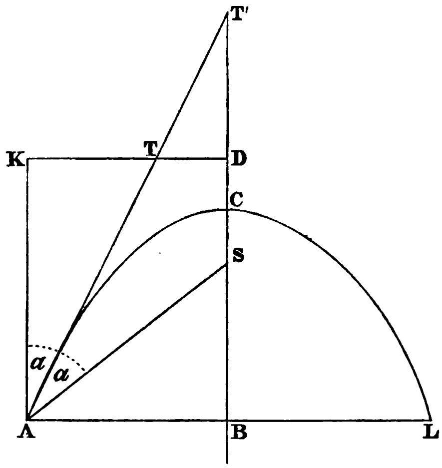
28. *To find the position and magnitude of this parabola.*

Through the point A draw a line AS making an angle $= \alpha$, with AT and in it take S such that $AS = \dfrac{V^2}{2g}$.

Then by the result of the last article, S is the focus of the required parabola.

In the vertical line through A take the point K such that $AK = AS$: draw AB, KD perpendicular to AK, and therefore horizontal; and through S draw BSD parallel to AK,

cutting the parabola in C: it is evident that BSD is the axis



of the parabola, and C its vertex bisecting SD; KD is its directrix.

We then get immediately from the figure

$$\begin{aligned} SC &= \tfrac{1}{2} SD \\ &= \tfrac{1}{2}(BD - BS) \\ &= \tfrac{1}{2}(AK - AS\cos 2\alpha) \\ &= \frac{1}{2}\left(\frac{V^2}{2g} - \frac{V^2}{2g}\cos 2\alpha\right) = \frac{V^2}{2g}\sin^2\alpha. \end{aligned}$$

If we represent the latus rectum by L,

$$\begin{aligned} L &= 4.SC \\ &= \frac{2V^2}{g}\sin^2\alpha. \end{aligned}$$

If h represent the greatest height above the horizontal plane through A to which the projectile ascends,

$$\begin{aligned} h &= BD - CD \\ &= \frac{V^2}{2g} - \frac{V^2}{2g}\sin^2\alpha \\ &= \frac{V^2}{2g}\cos^2\alpha. \end{aligned}$$

If AL be the horizontal range $= k$,

$$k = 2AB = 2AS \sin 2\alpha$$

$$= \frac{V^2}{g} \sin 2\alpha.$$

By reference to figure of Art. (27) we see that t', the time which the particle takes to pass from A to P, is the same as the time in which it would, under the action of gravity alone, fall from A to R, or, which is the same thing, from N to P; therefore $$NP = \tfrac{1}{2}gt^2,$$

therefore $$t = \sqrt{\left(\frac{2NP}{g}\right)}.$$

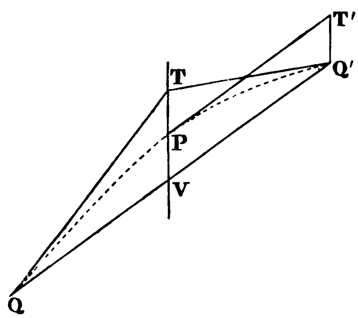
If therefore T be the time of the particle's moving from the point A to the vertex C of the parabola (fig. Art. 28.),

$$T = \sqrt{\left(\frac{2T'C}{g}\right)}$$

$$= \sqrt{\left(\frac{2BC}{g}\right)}, \text{ by a property of the parabola,}$$

$$= \frac{V}{g} \cos \alpha.$$

29. Let QQ' be any portion of a projectile's path; V the



middle point of the cord QQ'; P the point where the diameter PV meets the parabola; the particle describes the portions QP and PQ' in equal times.

To prove this, let T be the point in PV where the tangents at Q and Q' meet; also draw $Q'T'$ vertical and therefore parallel to PV, to meet the tangent at P in T'. We have seen that the time of the particle's moving from Q to P equals that of its falling by gravity alone from T to P; and similarly its time from P to Q' equal time of its falling from T' to Q'. Now by property of the parabola PT' is parallel to QQ', and also $TP = PV$,

therefore $$T'Q' = PV = TP.$$

Hence the time of falling through each of these spaces is the same, and the proposition is proved.

If therefore T represent the time of passing from Q to Q', T equal twice the time of falling through PV

$$= 2\sqrt{\left(\frac{2PV}{g}\right)}.$$

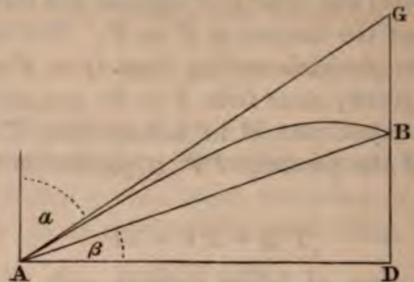
It is easily seen that this form immediately gives the value of T already obtained for the time between A and the vertex.

Again, the time from Q to Q' equal twice the time from P to Q', but the time from P to Q' is the same as the time in which the particle would have described PT' under the influence of the velocity alone which it had at P, equal $\frac{1}{2}$ the time in which the projectile would have described QQ' with the same velocity; hence the time from Q to Q' is the same as the time in which the particle would have described the chord QQ' with the aforesaid velocity. This result may be thus enunciated: A projectile describes any arc of its parabola in the same time in which it would describe a distance equal to the chord of that arc, if it moved uniformly with a velocity equal to that which it really has at that point of the arc where the tangent is parallel to the chord.

30. The following formula for determining the point where the course of a projectile meets a given line passing through the point of projection is often convenient.

Suppose the particle to be projected from the point A with a velocity V in the direction AG.

Let α be the inclination of AG to the vertical, B the



point when the curve meets the given line AB, and β the inclination of AB to the horizon. Draw GBD vertical and therefore parallel to the axis of the parabola: then by a property of the curve, if AS be the distance of A from the focus,

$$AG^2 = 4AS.BG.$$

Now $$AS = \frac{V^2}{2g},$$

and from the figure $$AG = AB\frac{\cos\beta}{\sin\alpha},$$

$$BG = AB\frac{\cos(\alpha+\beta)}{\sin\alpha};$$

therefore by substitution,

$$AB\frac{\cos^2\beta}{\sin\alpha} = \frac{2V^2}{g}\cos(\alpha+\beta),$$

therefore $$AB = \frac{2V^2}{g}\frac{\cos(\alpha+\beta)\sin\alpha}{\cos^2\beta}.$$

Also if T be time of passing from A to B,

$$T = \frac{AG}{V} = \frac{AB}{V}\frac{\cos\beta}{\sin\alpha}$$

$$= \frac{2V}{g}\frac{\cos(\alpha+\beta)}{\cos\beta}.$$

If the initial direction of the projectile had made an angle α with the line AB instead of the vertical, and if in this case B' had been the point where the parabola described by the

projectile met the line AB, we should get the value of AB' from the above expression for AB by putting $\left\{\frac{\pi}{2} - (\alpha + \beta)\right\}$ instead of α: but the result of this substitution gives

$$AB' = \frac{2V^2}{g} \frac{\cos\left[\left\{\frac{\pi}{2} - (\alpha + \beta)\right\} + \beta\right] \sin\left\{\frac{\pi}{2} - (\alpha + \beta)\right\}}{\cos^2\beta}$$

$$= \frac{2V^2}{g} \frac{\sin\alpha \cos(\alpha + \beta)}{\cos^2\beta}$$

$$= AB;$$

which shews that B and B' coincide; or that there are two directions in which a particle may be projected with a given velocity from a point A, in order to strike a given point, as B, these two directions being equally inclined to the vertical and to AB.

31. It would be beyond our limits to attempt here to investigate the motion of a particle under the action of any more complicated system of forces than appears in the case of a projectile; but there is a theorem asserting in some degree a relation between forces and the nature of the path which they cause a particle to describe, which requires a full explanation; it gives rise to the definition of what is termed *Centrifugal Force*.

If a particle be in motion unacted upon by any external force, the First Law of Motion asserts that it will continue to move *uniformly* in the same straight line. If now a force be applied to it having the same line of action as that in which it is moving, it will still continue to move in the same straight line but with a *varying* velocity according to the nature of the force; but if a finite force at right angles to its direction be alone applied to it, it will begin to alter the *direction* of its motion, the velocity manifestly remaining the same.

If both these forces act simultaneously upon the particle, they will by the Second Law of Motion be equivalent to a single force acting on the particle obliquely to the direction

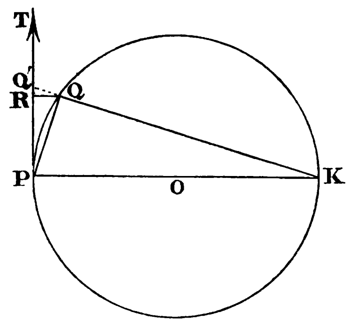
of its motion, whose effect will be to alter both the velocity and direction of the particle's motion.

Hence we not only see that a particle describing a *curve* must at every instant be under the action of some forces which are not coincident in direction with its motion, but also that if all the forces acting upon it at that instant be resolved into two, one in the direction of motion and the other perpendicular to it, the sole effect of this last is to make the particle move in the curve instead of in the straight line which is the *tangent* to the curve at the position of the particle; *i.e.* it draws the particle in the next instant of time away from the tangent, and has therefore a proper dynamical effect: we may however speak of it as acting to keep the particle in the curve, and therefore as counteracting an imaginary tendency on the part of the particle to fly out of it in the direction of the normal. The supposed force necessary to produce this tendency on the particle is termed *Centrifugal Force.*

From this explanation it follows that centrifugal force is not a real existent force acting upon the particle, but merely a statical measure in an opposite direction of the resolved normal force acting upon the particle, whose office is to draw the particle into the particular curve which it is describing. It is convenient to call this normal force the *Centripetal* force acting upon the particle, and then the *Centrifugal* force may be defined to be a force equal and opposite to the *Centripetal.*

32. We can illustrate what has been said by the example of a particle moving *uniformly* in a circle: of course in such a case the only force acting would be the centripetal force, which might be provided by the tension of a string, as when a ball is whirled round horizontally, or might be the resultant of any other assigned forces. Suppose that when the particle comes to a point P, the force or forces which act upon it suddenly cease; it would then, by the First Law of Motion, continue to move in the tangent PT to the circle with the velocity which it had at P: let Q' be the point at which

it would thus arrive in a small time t''. Let now Q be the point in the circle where it really does arrive in the same time; then it is clear that the effect of the centripetal force



in that time has been to draw the particle through a distance $Q'Q$: but according to the explanation above given, the effect of the centripetal force may be otherwise looked upon as serving to keep the particle at Q in opposition to that of the imaginary centrifugal force, which tends to draw the particle from Q back to the tangent.

It is not difficult to calculate the centripetal force necessary to act upon the particle at every point P of the circle in order to make it describe the curve uniformly with the given velocity v.

While the particle passes from P to Q, the direction of the centripetal force acting upon it is continually changing, for it must always be normal to the circle and therefore pass through the centre O: but if we take Q sufficiently near to P, we may suppose that for the very small arc PQ the centripetal force remains parallel to itself and of the same intensity as at P; in other words, that the arc PQ is described by the particle under the action of a velocity which would of itself carry it uniformly along PT, and a constant force perpendicular to PT equal to the centripetal force at P. Hence if we draw QR perpendicular to PT, by the Second Law of Motion (see Art. 12), RQ is the distance through which the particle would fall from rest under the

action of the centripetal force at P alone, in the same time as that in which it would describe PR with a uniform velocity v.

If t be this time, and f the measure of the accelerating force which the centripetal force exerts upon the particle, we thus get

$$PR = vt \qquad QR = \tfrac{1}{2}ft^2,$$

and therefore
$$f = v^2 \frac{2QR}{PR^2}.$$

Draw the diameter PK and join QK, QP; then from the similar triangles PQK, PRQ,

$$QR : QP :: QP : PK,$$

therefore by substitution

$$f = \frac{v^2}{\frac{1}{2}PK}\left(\frac{PQ}{PR}\right)^2.$$

But by Newton (*Lemma* VII. § 1.) as Q moves up indefinitely close to P, the ratio of PQ to PR approximates to and ultimately becomes unity; hence, as our hypothesis becomes justified only when Q is indefinitely near to P, we have, putting r for the radius of the circle,

$$f = \frac{v^2}{r}.$$

33. Since a very small arc taken at any point in a curve is coincident with the arc of the circle of curvature at that point, and its deflection from the common tangent at that point the same, it will follow that a particle moving in that curve and having a velocity v at the proposed point, is for the instant in the same circumstances as if it were describing the circle of curvature at that point with a uniform velocity v. Hence, putting ρ for the radius of this circle of curvature, the normal accelerating force acting upon the particle at this point of the curve must be $\frac{v^2}{\rho}$; and therefore this must also represent the accelerating centrifugal force upon the particle, estimated as the force which would counteract the accelerating normal force.

34. It is sufficiently clear from the foregoing explanations that in the case of a particle moving uniformly in a circle, a force equal to the centrifugal force would, if applied to the same point with the other forces which produce the motion, preserve equilibrium. Hence many problems of this nature may be solved as statical problems: this is an artifice of general application in the higher parts of Dynamics.

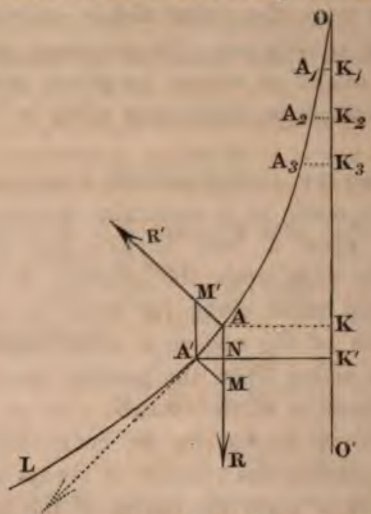
35. Sometimes we find a particle constrained by various circumstances to move in a particular course only. Thus a particle placed upon a hard surface and acted upon by gravity can only slide upon the surface; if within a small tube, it can only move along the tube: in such cases just so much reaction of the surface is called forth as will, with the other forces acting upon the particle, produce a resolved force in the direction of the normal able to draw the particle into the curve formed by the constraining surfaces: if the surfaces be smooth this reaction is always normal to them, and its sole effect is upon the course of the particle; it does not alter its velocity; the alteration that does take place in the velocity is therefore wholly due to the action of the other forces.

36. In the following simple case the law of the alteration of the particle's velocity is easily obtained.

Suppose the particle to be constrained to move along the smooth curve OL, the only force acting upon it, besides the normal resistance of the curve at every point, being a force which is constant and always parallel to OO': then the *increase* of velocity attained by the particle in passing from O to any point A in the curve, is the same as it would acquire if it were drawn from O by the force alone, which is parallel to OO', to a point K, defined by letting fall a perpendicular from A upon OO'.

For, take A' the point in the curve immediately succeeding A, so that AA' produced may be conceived to be the tangent at A. Draw AR parallel to OO', and AR' perpendicular to

AA'; these are respectively the directions of the forces acting upon the particle at A; their resultant, that part of the normal force which produces the deflection being omitted, is by question



in the direction of the particle's motion, *i.e.* along AA': if then the parallelogram $A'M'AM$ be completed, AM', AM, and AA' are respectively proportional to the normal force upon the particle at A, the force parallel to OO', and the resultant of these two, which produces the alteration in the particle's velocity: let the force along OK and the resultant along AA' be represented by f and f' respectively, then

$$f' : f :: AA' : AM \text{ .... } (1).$$

Let v be the velocity which the particle had at A, v' that which it has at A'; if then we suppose f' to be constant while the particle moves from A to A', as we may since AA' is very small,

$$v'^2 = v^2 + 2f'AA', \quad \text{Art. (25)}$$

$$= v^2 + 2f\,\frac{AA'^2}{AM}, \text{ from (1):}$$

but if N be the point where the perpendicular $A'K'$ meets AR, by similar triangles

$$AA'^2 = AM.AN;$$

therefore $$v'^2 = v^2 + 2fAN:$$

or drawing AK parallel to $A'K'$,

$$v'^2 = v^2 + 2f.KK'.$$

Similarly, if $A_1A_2A_3 \ldots\ldots A_nA$ be points in the curve indefinitely near to each other, A_1K_1, $A_2K_2 \ldots\ldots A_nK_n$, AK perpendiculars from them upon OO', v_1, $v_2 \ldots\ldots v$ the velocity of the particle when at those points respectively, V its velocity when at O, we have the following series of equations,

$$\left.\begin{aligned} v_1^2 &= V^2 + 2fOK_1, \\ v_2^2 &= v_1^2 + 2fK_1K_2, \\ v_3^2 &= v_2^2 + 2fK_2K_3, \\ \&c. &= \&c. \\ v_n^2 &= v^2_{n-1} + 2fK_{n-1}K_n, \\ v^2 &= v_n^2 + 2fK_nK, \end{aligned}\right\} \ldots\ldots\ldots\ldots (A);$$

therefore adding,

$$\begin{aligned} v^2 &= V^2 + 2f(OK_1 + K_1K_2 + \ldots\ldots + K_nK) \\ &= V^2 + 2fOK. \end{aligned}$$

We see therefore, that *the square of the velocity which the particle has at* A *is equal to the square of the velocity which it had at* O, *increased by the square of the velocity which the force* f *would alone generate in it, in drawing it from rest at* O *to* K.

Had the force f acted in the opposite direction, our equation would have been

$$v^2 = V^2 - 2fOK;$$

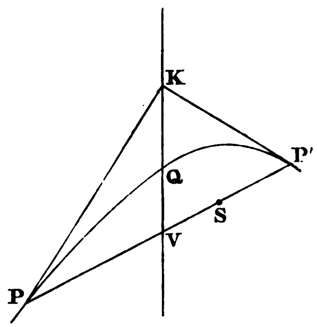
hence in this case we must put *diminished* for *increased* in the above assertion.

A particular case of this kind of motion is that wherein a particle slides down a smooth surface under the action of gravity; by the above form we see that the alteration in its velocity is exactly the same as would be produced if it fell through the same vertical distance under the action of gravity alone.

Examples to Section III.

(1). The time of a particle, under the action of gravity, describing any arc of its parabolic path, bounded by a focal chord, is equal to the time of falling from rest vertically through a distance equal to the length of that chord.

Let PSP' be any such chord, and let the tangents at P



and P' meet its diameter KQV in K; then, by the property of the curve, K is in the directrix and $KQ = QV$.

Now the time of describing PQP' equals twice the time of falling by gravity from K to Q, (Art. 29),

$$= 2\sqrt{\left(\frac{2KQ}{g}\right)}$$

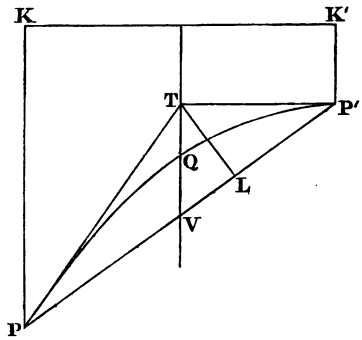
$$= \sqrt{\left(2\,\frac{2KV}{g}\right)}$$

$$= \sqrt{\left(\frac{2PP'}{g}\right)}.$$

But this last ratio is the time of describing the length PP' from rest, under the action of gravity; therefore the proposition is proved.

(2). The velocities at the extremities of any chord of the parabola described by a projectile, when resolved in a direction perpendicular to the chord, are equal.

Let PP' be any such chord, T the point in its diameter TQV where the tangents at P and P' intersect: draw PK,



$P'K'$ perpendiculars to the direction KK'. Then the velocities at P and P' are those due to falling by gravity through the lengths KP, $K'P'$ respectively.

We have, by a known property of the curve,

$$PT^2 = 4PK \cdot TQ,$$

and $$P'T^2 = 4P'K' \cdot TQ;$$

therefore $$PT^2 : P'T^2 :: PK : P'K'$$

$$:: (\text{velocity at } P)^2 : (\text{velocity at } P')^2.$$

Therefore PT and $P'T$, are respectively proportional to the velocities at P and P': and if TL be drawn perpendicular to PP' it must be proportional to the resolved part of either of them perpendicular to the chord; which proves the proposition.

(3). A body is projected with a velocity v in a direction making an angle β with the horizontal plane from the deck of a ship which is sailing in a direct course with a uniform velocity u, and hits a mark in the wake of the ship and in the plane of the deck at a given distance from the point of projection. Shew that if α be the angle of projection at which the body propelled by the same force would hit the same

mark had the vessel been at rest, then

$$\frac{v}{u} = \frac{2 \sin\beta}{\sin 2\beta - \sin 2\alpha}.$$

Let a be the given distance of the mark, t the time in which it reaches the mark; then, in the first case, since its horizontal velocity is $v \cos\beta - u$, and its vertical is $v \sin\beta$, we have, by resolution of motions (Art. 13),

$$a = (v \cos\beta - u)\, t,$$

and $$gt = 2v \sin\beta;$$

therefore $$a = (v \cos\beta - u)\frac{2v}{g}\sin\beta.$$

Similarly, we should get in the second case

$$a = v \cos\alpha \frac{2v}{g}\sin\alpha;$$

therefore $$(v \cos\beta - u)\sin\beta = v \cos\alpha \sin\alpha,$$

therefore $$\frac{v}{u} = \frac{\sin\beta}{\cos\beta \sin\beta - \cos\alpha \sin\alpha} = \frac{2 \sin\beta}{\sin 2\beta - \sin 2\alpha}.$$

SECTION IV.

MOTION OF A BODY.

37. In the foregoing investigations regarding the movement of a particle, we have obtained formulæ connecting together the symbols which represent the numerical measures of the various quantities, space, time, velocity, and force: by the aid of these formulæ, if a sufficient number of these quantities be given in any proposed case, the others may be found by simple analytical processes. But it is important here to recollect, that the measure of force involved in these formulæ is the measure of the *accelerating force* exerted upon the *particular particle* under consideration. As then (Art. 7) the measure of any given force, estimated as an *accelerating force*, varies with the mass of the different particles to which it is applied, it becomes necessary for us, before we can use our formulæ generally, to investigate the relations which hold between the masses of particles, forces applied to them, and the accelerating effects which they produce.

38. The meaning of the term *Mass* has been already (Art. 7) explained: we must now define how it is measured.

DEF. *The masses of bodies, considered as material points, are proportional to the forces which will generate in them respectively the same velocities in the same duration of time.*

This definition of the measure of mass, like that of the measure of accelerating force, is not abstract; it requires to be justified by experiment: for supposing two bodies to coalesce, whose masses are equal by the above definition, that is, two bodies in which equal forces would generate the same velocities in the same time; the body formed by this combination would,

of course, have a mass equal to twice the mass of either of the bodies so combined, and ought therefore, in order to be true to the definition, to require twice as great a force as either of them to generate a given velocity in it in a given time: but that this is really the case can only be known as the result of experiment.*

39. If we define *the momentum of a body at any time to be the product of its mass into the velocity which it has at that time*, the results of the experiments alluded to can be easily enunciated in the form which is known as,

The Third Law of Motion. Forces are proportional to the momentum which they generate in any mass in the same time.

Hence, if a force represented by F generate a velocity V in a mass M in a *unit* of time, we have

$$F = CMV,$$

where C is some constant whose value depends upon the convention which we make with regard to the units of force, mass, and velocity.

It is convenient to assume these units, so that $C = 1$, and therefore

$$F = MV.$$

This makes $M = 1$, when F and V are put each equal to 1; which implies that the unit of mass is the mass of that body in which a unit of force will generate a unit of velocity in a unit of time.

40. Hence, upon the usual supposition that a second is the unit of time, that the velocity which would carry a particle over a foot in one second is the unit of velocity, and that the force which the earth's attraction exerts dynamically upon a pound weight of matter is the unit of moving force, the unit of mass would be that in which a force equal to a pound weight would in one second generate a velocity such as would alone carry it over a foot space in the next second.

* See next Section.

41. In the formula

$$F = MV,$$

V is the velocity which the force generates in 1″, in the body whose mass is M; it is therefore, according to Art. (6), the measure of the *accelerating force* which F exerts upon M; F itself is called the measure of the *moving force*. This accelerating force may very well, in consistency with the symmetry of our notation, be represented by f, so that

$$F = Mf;$$

therefore

$$f = \frac{F}{M}.$$

It is now evident, that in investigating the motion of a material particle, or of a body which we can consider, as regards its motion, to be condensed into one point, we shall obtain the accelerating force required for our formula, by dividing the measure of the whole moving force applied to the mass by the measure of the mass which is moved by it.

This principle is illustrated by the examples which follow.

42. Before however entering upon them, it may be as well to explain the nature of one force, that of gravity, which plays an important part in most of them. We have already become acquainted with it as a statical force, and have denominated it weight. It has been conveniently represented by the symbol W.

Upon considering it with reference to Dynamics, we find that all bodies submitted *in vacuo* at the same place to its free and unrestricted action acquire the same velocity in the same interval of time, whatever may be their individual magnitudes or natures; we must therefore conclude that the whole force, *i.e.* the moving force which gravity exerts upon any body, is always proportional to the mass of that body. (Art. 41).

It is also found that the velocity which a body, falling from rest under the action of gravity alone, acquires, is always proportional to the square of the time of falling: but this result is only true of a body moving under the action of a

constant force (Art. 18); hence the accelerating force of gravity is the same for all bodies at the same place; it is usually represented by g; its numerical value has been already explained, Art. (14).

Hence we have by Art. (39), if W' be the moving force of gravity exerted upon a mass M,

$$W' = Mg.$$

Of course, in consequence of the convention of Art. (40) with regard to the unit of mass, W' is estimated in terms of the *moving force* which gravity exerts upon a pound weight of matter: now the Second Law of Motion asserts that there is no difference in the intensity of any force, and therefore of gravity, whether it be acting upon a body at rest or upon a body in motion, and therefore for W' we may put W; hence

$$W = Mg,$$

where W is the statical measure of the force which gravity exerts upon a mass M.

43. The remark that was made in Art. (9) respecting the accelerating force of impulses, is equally applicable to their moving force. Impulsive forces, like all finite forces, are proportional to the momentum which they generate in a given time; but as their time of action is by definition inconceivably short, and may therefore be considered the same for all of them, they are measured by the momentum which they have generated at the end of the time of their action.

44. In the elementary part of our subject we are but little concerned with impulsive forces, except as they arise from the impinging of one body upon another: but before we proceed to consider problems of this class we must become acquainted with two important experimental facts.

The *first* of these is included in the assertion that *Action and Reaction are equal and opposite.* We have already employed this principle very extensively in Statics. When two surfaces were supposed to be in contact, we assumed that their

mutual pressure was the same for both in intensity, but opposite in direction. The hand that presses upon the table is equally pressed back by the table: the stone that lying on a pumpkin crushes it with its weight, is acted upon by an equal force in an upward direction: the string or rod which transmits a force from one body to another, is only a medium by which the same force is exerted upon both bodies in an opposite direction. And generally, by whatever means, whether by connecting rods, strings, or by invisible and immaterial bonds, as in attraction, one body exerts a force upon another, the second exerts the same force in an opposite direction upon the first. Experiment leads us to the conclusion that this principle is true, as well in Dynamics as Statics, and equally so in regard to impulsive as to finite forces.

The *second* may be stated thus: If two spherical bodies moving uniformly in the same straight line approach each other so as to impinge the one upon the other, immediately after the impact they will recede from each other: it is always found that the velocity with which the two bodies recede from each other after impact, bears to the velocity with which they approached before impact a ratio which is constant for bodies of the same substance and independent of the actual velocities of the bodies. This ratio is called the modulus of elasticity of the bodies, and is never greater than 1: when it equals 1 the bodies are said to be perfectly elastic, when less than 1 they are imperfectly elastic.

45. If the two bodies be not moving in the same straight line, we may resolve their respective velocities into two parts, one along the common tangent to the two bodies at the point where they touch, and the other perpendicular to it; then the resolved velocities perpendicular to the common tangent follow the law just enunciated, while the resolved velocities along the tangent remain unchanged, if the bodies be smooth.

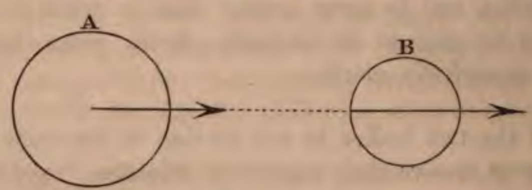
46. These results may easily be accounted for by supposing the impact of the two bodies to be separated into two parts:

after the two surfaces have come in contact the centres of the spheres will still approach each other for a short period, and thus the two bodies will be compressed; the mutual force which they exert upon each other at this time may be called the force of Compression; as soon as this force has reduced the velocities of the centres to an equality so that they no longer approach each other, the elasticity of the bodies will then cause them to strive to return to their original shape, and thus produce a force tending to separate them, which may be called the force of Restitution. If we assume that in the same bodies the force of restitution bears to that of compression a constant ratio independently of the initial velocities of the bodies, we shall upon investigation discover that the relative velocities of the spheres before and after impact are connected by the relation which was enunciated above.

47. The foregoing data enable us to solve the following general proposition respecting the impact of two spheres.

Two spheres A and B, whose masses are M and M' respectively, and elasticity e, are moving in the same straight line with velocities V and V', required their velocities after impact.

We will suppose them to be both moving in the same



direction, that A is the hindmost sphere, and that V is greater than V'. Let v and v' be their respective velocities after impact.

By the principle that action and reaction are equal, the same force acts both upon A and B at the instant of impact, it generates momentum in B because it accelerates its motion, but it destroys momentum in A; therefore, since the time of the action of this force is identical for both bodies, we must

have the momentum generated during the impact in B, must equal that destroyed in A, and therefore

$$M(V - v) = M'(v' - V'),$$

or

$$Mv + M'v' = MV + M'V' \text{..........} \quad \text{(I)}.$$

This equation asserts that the sum of the momenta of the two bodies is the same both before and after impact. It is not difficult to see that this is only another form of declaring the principle of the equality of action and reaction.

Again, from the definition of elasticity we have

$$v' - v = e(V - V') \text{............} \quad \text{(II)}.$$

Between (I) and (II) we get immediately

$$\left.\begin{aligned} v &= \frac{MV + M'V' - M'e(V - V')}{M + M'} \\ v' &= \frac{MV + M'V' + Me(V - V')}{M + M'} \end{aligned}\right\} \text{......} \quad (a),$$

which gives the velocities required.

48. If B had been moving towards A in the first instance, the numerical value of its velocity being as before equal to V', the total momentum of the system would have evidently been $MV - M'V'$, and the relative velocities of A and B, $V + V'$; hence this case would be derivable from (I) and (II) by affecting V' with a negative sign; and generally, if velocity in one direction be considered positive, that in the opposite direction will be correctly indicated by the negative sign.

49. If B were fixed, *i.e.* if A impinged upon a fixed obstacle, we should have directly from (II), since in that case both V' and v' must equal zero,

$$v = -eV,$$

that is, A would bound back from B with a velocity equal e times that with which it approached B.

This same result can be obtained from (a), for putting V' equal 0, we must suppose B to remain at rest after impact,

because its mass is so great with respect to that of A, that the velocity generated in it is inappreciable; this supposition would give us

$$\frac{M}{M'} = 0, \qquad \text{and therefore from } (a)$$

$$v = -eV,$$

$$v' = 0.$$

If the impinging bodies were inelastic they would not, by definition, separate after impact; indeed in that case, since $e = 0$, equations (a) give us

$$v = \frac{MV + M'V'}{M + M'} = v'.$$

Examples to Section IV.

(1). Two weights of 5 and 7 lbs. respectively are connected by a string which passes over a smooth pulley: find in pounds the tension of the string.

Let M and M' be the masses of the two weights, so that Mg and $M'g$ are the moving forces of 5 and 7 lbs. respectively: let also T be the moving force of the tension of the string.

Then the accelerating force acting upwards upon the 5 lbs. weight must be $= \dfrac{T - Mg}{M}$; and the accelerating force upon the 7 lbs. weight acting downwards must be $= \dfrac{M'g - T}{M'}$.

Now since these bodies are connected by a string of constant length, the velocities which they acquire in any time must be the same for both, in an upward direction for the first, in a downward for the second: therefore the accelerating force upon each is the same;

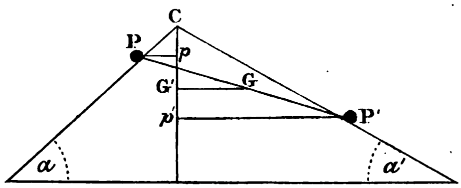
therefore $$\frac{T - Mg}{M} = \frac{M'g - T}{M'};$$

$$T = \frac{2MM'g}{M + M'} = 2\frac{M}{M + M'}M'g = 2\frac{5}{5+7} \text{ (moving force of 7 lbs.)},$$

therefore the tension $= \frac{10}{12}$ of 7 lbs. $= \frac{35}{6}$ lbs.

(2). Two equal bodies connected by a string are placed upon two planes which are inclined at angles α, α' to the horizon and have a common altitude: prove that their centre of gravity will move, as to a vertical direction, as if acted upon by an accelerating force $= g \sin^2 \frac{\alpha - \alpha'}{2} \cos^2 \frac{\alpha + \alpha'}{2}$.

Let PP' be the masses of the bodies, G the position of



their centre of gravity at any time; Pp, $P'p'$, GG' perpendiculars drawn upon the vertical through C the common vertex of the planes.

Then since $PG : P'G :: P' : P$, we get

$$CG'(P + P') = Cp.P + Cp'.P'$$
$$= CP \sin\alpha.P + CP' \sin\alpha'.P'.$$

Now it can be easily shewn, as in the last example, that the accelerating force upon both P and P' is $= \frac{P \sin\alpha - P' \sin\alpha'}{P + P'} g$; therefore if l were the value of CP when the bodies started from rest, and t the time since that instant,

$$CP = l + \frac{1}{2}\frac{P \sin\alpha - P' \sin\alpha'}{P + P'} gt^2.$$

Similarly, if l' were the original value of CP',

$$CP' = l' - \frac{1}{2}\frac{P \sin\alpha - P' \sin\alpha'}{P + P'} gt^2;$$

therefore, by substitution above, we get

$$CG' = \frac{l \sin \alpha . P + l' \sin \alpha' . P'}{P + P'} + \frac{1}{2} \left(\frac{P \sin \alpha - P' \sin \alpha'}{P + P'} \right)^2 g t^2.$$

Now this is exactly the formula for the motion of a particle in the vertical direction under the action of an accelerating force $\left(\frac{P \sin \alpha - P' \sin \alpha'}{P + P'} \right)^2 g$.

If P equal P', this becomes $\left(\frac{\sin \alpha - \sin \alpha'}{2} \right)^2 g$

$$= \sin^2 \frac{\alpha - \alpha'}{2} \cos^2 \frac{\alpha + \alpha'}{2} g.$$

(3). An imperfectly elastic ball is projected at an angle of 45° against a smooth vertical wall, the motion taking place in a plane perpendicular to the wall: after impact the ball strikes the ground between the wall and the point of projection, and rebounding once more reaches the point of projection.

Shew that if a be the distance of the point of projection from the wall, and b the height above the ground of the point where the ball strikes the wall, then

$$b = a \frac{1 - ee'}{1 + e};$$

where e is the ratio of the elasticities of the ball and wall, and e' that of the ball and ground, which is supposed perfectly smooth.

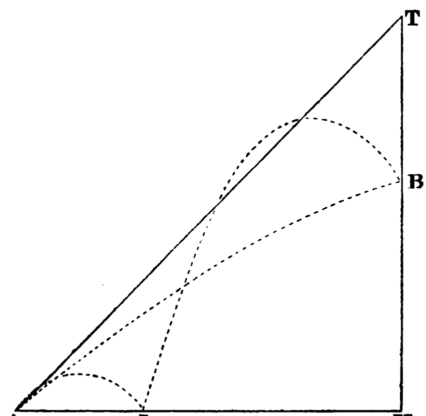
In the annexed figure the dotted curve AC, CB, BA represents the path of the particle.

Let AT be the direction of projection making an angle α with the horizontal plane ACK; also let V be the velocity of projection: by the question $AK = a$, $KB = b$.

Upon the balls rebounding at B, the velocity resolved vertically remains unaltered, but the part resolved horizontally is changed in the ratio of $e : 1$. (Arts. 45, 49.)

Also at C the vertical velocity is changed in the ratio of

$e' : 1$, but the horizontal velocity is the same as it was after rebounding from B.



Let t be the whole time between starting and returning to A; then t is the time in which the ball would traverse AK with the resolved horizontal velocity at A continued uniform, together with the time in which it would return over the same space with the resolved horizontal velocity, after rebounding from B;

therefore $$t = \frac{a}{V \cos} + \frac{a}{eV \cos \alpha}$$

$$= \frac{a}{V \cos \alpha} \frac{1+e}{e}.$$

Again, in this time t the body ascends to, and descends from the highest point of the parabola BC; it also ascends to, and descends from the highest point of the parabola CA: in other words, in the time t, gravity destroys and generates the vertical velocity $V \sin \alpha$, while the body describes BC; it also destroys and generates the vertical velocity $e'V \sin \alpha$ in the curve CA; therefore

$$t = 2\frac{V \sin \alpha}{g} + 2\frac{e'V \sin \alpha}{g} = 2\frac{V \sin \alpha}{g}(1 + e');$$

therefore, equating these two values of t, we get

$$\frac{a}{V\cos\alpha}\,\frac{1+e}{e} = 2\,\frac{V\sin\alpha}{g}\,(1+e'),$$

or $$2\,\frac{V^2}{g}\sin\alpha\cos\alpha = a\,\frac{1+e}{e(1+e')}\,\ldots\ldots\ldots(1).$$

Again, if S were the focus of the parabola AB,

$$AT^2 = 4AS.TB,$$

or $$\frac{AK^2}{\cos\alpha^2} = \frac{2V^2}{g}\,(AK\tan\alpha - b),$$

or $$2\,\frac{V^2}{g} = \frac{a^2}{\cos^2\alpha\,(a\tan\alpha - b)}\,\ldots\ldots\ldots(2).$$

Substituting in (1) and putting $\alpha = 45°$, we find

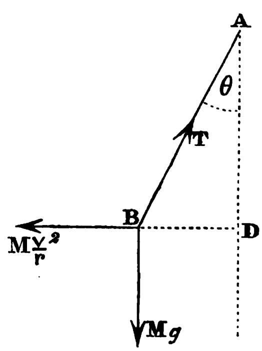
$$\frac{a^2}{a-b} = a\,\frac{1+e}{e(1+e')},$$

or $$a(e+ee') = (a-b)(1+e);$$

therefore $$b = a\,\frac{1-ee'}{1+e}.$$

(4). A stone weighing $\frac{1}{4}$lb. attached to the end of a string 3 feet in length, of which the other end is fixed, is made to revolve horizontally at the rate of three turns per second. Find the inclination of the chord, and its tension.

Let AB be the string, length l, B the stone whose mass is M, and v the velocity with which it is revolving.



Draw BD horizontal meeting the vertical through A in D.

Since the body is moving in a horizontal circle whose radius is BD, $= r$ suppose, under the action of the tension T of the string BA, and its own weight Mg, these two forces must be equivalent to a *centripetal accelerating* force upon B in the direction BD and of the magnitude $\frac{v^2}{r}$ (Art. 32).

Hence if we apply to B a moving centrifugal force $M\frac{v^2}{r}$ in the direction of DB produced, it would keep equilibrium with the forces T and Mg in their own directions.

Let BAD be represented by θ, then we have

$$\frac{T}{Mg} = \frac{1}{\cos\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{M\frac{v^2}{r}}{Mg} = \frac{\sin\theta}{\cos\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which gives $$\tan\theta = \frac{v^2}{rg}.$$

Now since the velocity is such that three times the circumference of the circle is described in a second,

therefore $$v = 3 \times \text{circumference}$$
$$= 6\pi r;$$

therefore $$\tan\theta = \frac{36\pi^2 r^2}{rg} = 36\pi^2\frac{r}{g}$$
$$= 36\pi^2\frac{l}{g}\sin\theta;$$

therefore $$\frac{1}{\cos\theta} = 36\pi^2\frac{l}{g} = 36\pi^2\frac{3}{32.2} \quad \ldots\ldots\ldots\ldots (3).$$

Hence from (1), since $T = \dfrac{1}{4\cos\theta}$ lbs.,

$$T = \frac{27\times\pi^2}{32.2}\text{ lbs.}$$
$$= 8.278 \text{ lbs.}$$

From (3) θ may be found to equal $88^\circ\ 16'$ nearly.

SECTION V.

PHYSICAL LAWS.

50. In the preceding pages we have made some of the various phenomena of Nature, such as force, space, velocity, &c., the subject of our definitions for the purpose of discovering the relations which exist between them under given circumstances: this may be said shortly to be the object of the science of Mechanics. In order to facilitate our investigations, we have represented the subjects of our definitions by algebraical symbols, and have determined formulæ connecting them, by a reference partly to these definitions, but chiefly to the nature of the subjects themselves; hence, since every definition necessarily takes for granted a certain amount of knowledge concerning the things defined, we may say, what is indeed a self-evident necessity, that our science is based entirely upon laws which must be deduced from direct observation of nature.

51. These laws have been distinctly enunciated at those points in the subject where their several applications first became necessary; but it may be useful here to give the following recapitulation of them.

(1). If a force act at any point A of a body, its point of application may be transferred to any point B, in its line of direction, without producing any alteration in its effect upon the equilibrium of the body, always provided that B be connected rigidly in any way with A, and that the direction of the force remain the same as at first.

(2). The action and reaction of bodies upon one another, however they may be transmitted, are, estimated as statical forces, equal and opposite.

(3). *First Law of Motion.* A material particle if at rest will continue at rest, and if in motion will continue to move uniformly in the same straight line unless it be acted upon by some external force.

(4). *Second Law of Motion.* When any number of forces act simultaneously upon a material particle in motion, the instantaneous quality of each force as regards its intensity and direction is the same as if it had acted alone upon the particle at rest.

(5). *Third Law of Motion.* Forces are proportional to the momentum which they generate in any mass in the same duration of time.

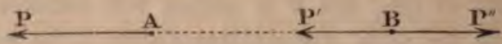
Of course it is manifest, that it is only to the facts which these laws assert that we are obliged to pay regard. The number of the laws which any writer may choose to recognize, and the form of their enunciation, are matters almost arbitrary with him; some of them he may embrace in his definitions, and others he may enunciate in such a way that the remainder shall follow from them as corollaries. Amongst English writers it has been generally considered most convenient to adopt the preceding arrangement of them; although even with them there has been lately a growing disposition to look upon the *Third Law of Motion* as a deduction from *First* and *Second Laws.*

52. We will now give a slight glimpse at the nature of the evidence upon which these laws are based.

As the law (1) has been already sufficiently treated of in Statics (Art. 15), it is not necessary to enlarge upon what was there said. It is as well however to remark, that it is sometimes included in the definition of the statical equality of forces: for instance, if it be defined that two forces are statically equal, when, upon being applied in the same straight

line but in opposite directions to two points respectively, which are in any manner rigidly connected together, they keep those points at rest, the above law would result as a theorem.

Thus, let the point of application of a force P be A, and



let B be any other point in its direction, rigidly connected in any manner with A.

At B apply simultaneously two forces P' and P'', both equal to P and in the line AB; and let P'''s direction be from and that of P' towards A. It is self-evident that the application of these forces cannot affect in any way the state of the system. Then, by definition, P'' and P will counteract each other, and therefore their simultaneous action has no effect upon the system, hence they may be removed. The only force then left will be P', equal to and in the same direction with P, but applied at B; and it has been proved that the system under its action is in the same condition as it was under the action of P at A. Q. E. D.

The meaning of the law (2) was explained in Art. (44). It is so often tacitly assumed in our consideration of the commonest mechanical arrangements that occur in every-day life, that its truth is almost self-evident, and any striking illustration of it rendered difficult. But it may be proved by the transmission of force from one body to another, through the medium of springs or elastic cords: the inspection of these will easily shew that the force transmitted to each body is the same.

The laws marked (3), (4), (5), are usually termed emphatically the *First*, *Second*, and *Third* laws of Motion.

It is obvious that the first of these could only be proved directly by the observation of the permanent state of a body when all external causes of motion have been removed from it: as this case can never be found in nature or produced artificially, we must be content with as near an approximation to it as possible. In truth, we can only ascertain that the more the external influences acting upon the body are re-

moved, the more nearly does its observed motion approach to that indicated by the enunciated law.

Thus if a ball be rolled along a gravel path, its velocity will rapidly diminish and it will soon stop altogether; if it be projected along a pavement, its motion will be longer in its duration and more uniform; if along a piece of ice the motion will continue still longer; and generally, the smoother the surface is upon which we roll it, the farther and the more uniformly will the ball move: we are justified in imagining from this, that if we could entirely remove friction and the resistance of the air, which are only forces acting upon the ball, it would roll on for ever with the velocity which it had initially. Again, on a horizontal line of railway it is found that a constant power must be exerted by the engine in order to maintain a train once in movement at a uniform speed; it would therefore seem that the engine serves only to counteract the constant retarding force of friction and of the air's resistance, and that then the velocity of the train continues uniform in obedience to our law. An instance of the practical use made of this law is afforded by the method which a railway labourer adopts for emptying ballast-waggons: the loaded waggon is impelled at considerable speed to the spot where its contents are to be deposited, it is then suddenly stopped, while at the some moment the front side of it is removed; there exists therefore nothing to stop the velocity of the loaded earth, which consequently continues its course without the waggon, and falls into the place for which it was destined. If a card be placed upon the tip of the finger, with a penny resting above it, it may be filliped away leaving the penny still resting upon the top of the finger. Instances of these kinds might be multiplied without number, and must be familiar to every person.

The law (4), or *Second Law of Motion*, asserts that the instantaneous action of a force upon a material particle is, as regards its intensity and direction, in no way altered by the existence of velocity in the particle, or by the simultaneous action of other forces. It is evident that this law can only

be illustrated by the exhibition, as facts, of some of the results to which it would theoretically lead us; such as the proportionality of accelerating forces to the velocities which they generate in a given time, and the law of composition of motions given in Art. (12). An instance of the first of these will be adduced when the nature of gravity is treated of. The second is of such universal occurrence, that it seems hardly possible that it should remain unnoticed by the most careless observer. A ball dropped from the window of a railway-carriage which is proceeding at full speed, will, when it reaches the ground, be just vertically below the window, although in the time during which it was falling the carriage must have advanced a very considerable distance: this shews that when the ball leaves the hand, necessarily impressed with the same horizontal velocity as that which the hand or the carriage possesses, it moves in a *forward direction*, as if it were influenced by that velocity only, while at the same time it falls *downwards*, as if gravity alone had acted upon it.

If from a point at a certain distance above the ground one ball be projected horizontally, while another is at the same instant dropped, it is found that they both strike the ground at the same instant.

A person walking steadily in a straight line will, if he throws a ball, as he thinks, straight upwards, find that it always falls into his hands again, and not behind him, as he might have expected that it would.

These examples are evidently in conformity with the principle, that if two causes act simultaneously upon a body for a given time, which would separately have carried it through certain linear spaces in that time, then, by their united action, the body in the same interval arrives at the extremity of the diagonal of the parallelogram described upon these lines as sides.

It is not easy to give simple illustrations of Law (5), or *Third Law of Motion*, other than those afforded by the use of Atwood's Machine (see Art. 56).

53. It has been already observed with regard to this law, that many mathematicians consider it so evident a deduction from the laws which precede it, that it ought not to be the subject of a separate enunciation, but should be more properly appended to them as a corollary. Their reasoning is somewhat of this nature.

Suppose a force represented by F to generate in a given time a velocity v in a mass M; also in the same time let a force F' generate a velocity v' in a mass M': F and F' are estimated upon the consideration that forces are equal when in a given time they generate the same velocity in the same material particle.

The velocity generated in M *is the same as would be generated in any one of its units of mass separated from it by a force equal to* $\frac{\mathrm{F}}{\mathrm{M}}$, *acting alone upon that unit for the given time.*

The same may be said with regard to a unit of mass separated from M'.

Hence, since the unit of mass is the same for both, these velocities must be proportional by the Second Law of Motion to the forces which generate them; therefore

$$v : v' :: \frac{F}{M} : \frac{F'}{M'},$$

or

$$\frac{F}{F'} = \frac{Mv}{M'v'};$$

and therefore

$$F \propto Mv.$$

The assumption here made is distinguished by being written in italics; it is simply the same thing as that which the definition of mass asserts, *i.e.* the strict proportionality between masses and the forces which generate in them the same velocity. But this measure is, as remarked in Art. (38), only justified by the Third Law of Motion, itself or some other assertion equivalent to it: in the same manner as the Second Law of Motion proves the propriety of our measure of accelerating

force, does the Third Law prove the correctness of our measure of mass: as, therefore, it does not seem that this law can be deduced from the preceding laws without an accurate idea of the measure of mass, it can only stand alone as an experimental fact, unless it be conceived that the measure of mass can be deduced from the Second Law of Motion.

54. It must not be imagined that the laws which have just been enunciated, as well as those of any other of the Physical sciences, depend much for their proof upon the evidence of direct experiment. Every natural phenomenon that we can observe is the result of many simultaneous causes which produce their effects in accordance with certain laws: it is almost impossible for us to isolate any particular cause and its effect, and so to ascertain immediately the law which connects them; our object can only be effected by various experiments, all of which shall include the same cause under different circumstances: a careful comparison of all the results may perhaps enable us to eliminate and ascertain the particular effect.

After all has been done that the most accurate experiments can effect, the apparent laws obtained can only be considered as suggestions of laws. They are then assumed to be true, and calculations of the consequences of certain physical data are made upon them as such; the results of these calculations are then compared with the results of actual observation of the same physical circumstances, and it is in the exact accordance of these two results that the proof of the truth of these laws consists.

Of all the Physical sciences that of Astronomy is by far the most complete, it may be said to be almost perfect: it not only accounts for every movement of the celestial bodies in the minutest particulars, but it boldly predicts phenomena which without its aid we could not have anticipated, and points out to us the existence of planetary bodies which observation alone had failed to detect: we cannot conceive a more convincing proof that the laws upon which this science is based are rigidly true. When then we say that these

laws are none other than those which we have just been discussing, we need not repeat that they rest upon a far surer foundation than experiments, of the nature of those to which we have alluded in our illustrations, could possibly afford.

55. Before we close this section, we will say a few words upon the experiments which have been made relating to the nature of the force of gravity, and which lead to conclusions already stated in (Art. 42).

The assertion that gravity impresses the same accelerating force upon every body which is acted upon by no other force, whether it be great or small and whatever be its density, may seem at first to be contrary to our daily experience: for we are in the habit of observing that some things fall to the ground from the same height much quicker than others; for instance, that if a penny and a feather be dropped at the same moment, the penny will reach the ground long before its companion; while indeed a soap-bubble or a balloon, when submitted to the action of gravity, actually rises instead of falling. It is not difficult however to account for these anomalies: an atmosphere exists everywhere around us, which offers resistance to the passage of any object through it; this resistance depends for its magnitude solely upon the velocity and the form of surface of the moving body, not at all upon its mass; and hence, in the case of two bodies which have the same form, since the moving force of the resistance would be the same for both, its *retarding effect* would be greater upon that body of the two which had the least mass, and therefore of course the heavier body would reach the ground first. The upward movement of the balloon or bubble depends upon a somewhat different principle, which it belongs to Hydrostatics to demonstrate; but still it is due to the action of the air.

We accordingly find, that if we allow bodies to fall by the action of gravity in a glass vessel which has been exhausted of air, they all fall alike and reach the ground at the same instant, whatever may be their individual densities. If, how-

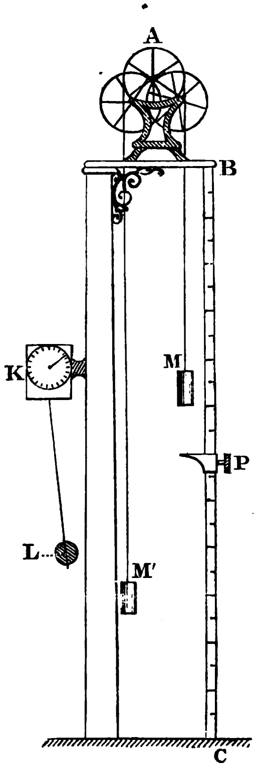
ever, the air be readmitted to the vessel, and the experiment then repeated, all the inequalities in the bodies falling reappear. Such experiments as these leave no doubt that the moving force of gravity upon different bodies is proportional to their masses.

In order to prove that the accelerating force of gravity, which has just been spoken of as being the *same* for all bodies, is also absolutely *constant*, we must make it evident that, when bodies move under its action alone, their velocities are proportional to the time, and the space described by them to the square of the time. Considerable difficulties stand in the way of doing this by direct experiment. In two seconds a body falls 64 feet, and in four seconds about 256 feet; in four seconds, too, it would acquire a velocity of 128 feet per second. Not only should we be liable to very great errors in noting such rapid movements, but the resistance of the air attendant upon them would be a very severe disturbing cause to the free action of gravity.

Various ingenious contrivances have been effected for diminishing the intensity of the force of gravity without altering its nature. Galileo suggested the use of a smooth inclined plane for this purpose: the accelerating force upon a body sliding down such a plane would, if we neglected the small amount of friction, bear to the accelerating force of gravity the ratio of $\sin \alpha : 1$, where α represents the inclination of the plane to the horizon; by diminishing α, this accelerating force could be made as small as we pleased, and still would bear a constant ratio to the accelerating force of gravity.

56. Of all contrivances of this kind Atwood's is by far the most complete. The annexed figure represents it. Two weights, whose masses are M and M', are connected by a very slight string, whose weight may be neglected: this string passes over a pulley A, whose pivots rest upon the circumferences of two friction-wheels. BC is graduated to mark the distance which M describes in its movements, and KL is a clock attached to the pillar which supports the wheels at A.

If M and M' are exactly equal, the whole system is in equilibrium; but if M is greater than M', it will fall by the action of gravity, and will draw M' upwards. We may consider the moving force which gravity exerts upon the difference between the masses of M and M', i. e. upon $M - M'$, to be the whole force acting upon the system, because its moving force upon the remaining masses destroy each other: but the whole mass thus put in motion by the weight of $M - M'$ is $M + M'$; therefore the accelerating force upon the whole system, and consequently upon M, is equal to



$$\frac{\text{the moving force of gravity upon } (M - M')}{M + M'}$$

$$= \frac{M - M'}{M + M'} g,$$

where g represents generally the accelerating force of gravity. Hence we are provided with the means of observing the movement of M under the action of an accelerating force $\frac{M - M'}{M + M'} g$, whose ratio to the accelerating force of gravity is a constant quantity, and is capable of being diminished *ad libitum*. Whatever can be ascertained by the observation of the movement of M under these circumstances, with regard to the law of variation of the force $\frac{M - M'}{M + M'} g$, must be true also for g; whilst the power of diminishing $\frac{M - M'}{M + M'} g$, and therefore of making M move as slow as we like under its action, renders the requisite observations comparatively easy.

It is convenient, in trying experiments with this machine, to make M heavier than M' by placing a small bar of metal

upon it; movement will then immediately commence, and will be the slower the smaller the bar is: but whatever be the size of the bar, it is always found that the spaces through which M descends in any times are proportional to the squares of those times.

If a ring P be screwed on to the graduated rod BC, of sufficient size to allow M to pass through it in its descent, but small enough to stop the bar which lies across M, the subsequent movement of M ought, by the First Law of Motion, to be uniform, and of the same velocity as that which it had at P; because, with the bar, all moving force upon M would be removed. This is found to be true, and thus forms another illustration of the First Law of Motion. The space that M describes in the first second of time after passing P is, by our definition (Art. 2), the measure of the velocity which it had at P, and which it had acquired by falling from rest to P. Wherever P be screwed, all experiments shew that the velocity acquired by M in falling from rest to P is proportional to the time which it occupies in so falling.

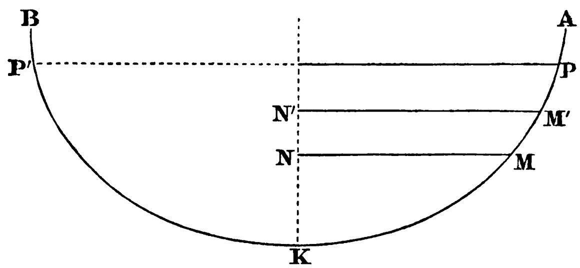
Both of these experimental results, with regard to the space described and the velocity acquired by M in falling during any given time from rest under the action of the accelerating force $\frac{M - M'}{M + M'} g$, lead to the conclusion that this force, and therefore that the accelerating force of gravity upon all matter at the same place, is *constant*.

57. It is easy to see that by this machine we can illustrate both the Second and Third Laws of Motion. Art. (52).

SECTION VI.

CYCLOIDAL OSCILLATIONS.

58. We have seen (Art. 36), that if a heavy particle slide



under the action of gravity down the smooth surface of a curve, as AKB, the increase which its velocity would experience in passing from M' to M is the same as would be acquired by the particle in falling freely from N' to N, the vertical distance between M' and M, under the action of gravity alone, the velocity in this case at N' being the same as that of the proposed particle at M'. And similarly, if the particle were moving upwards from M to M', its velocity would be diminished according to the same law.

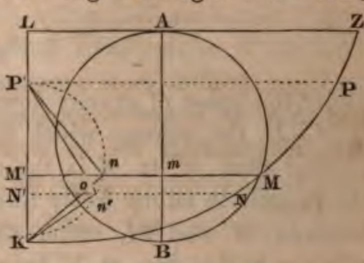
If, therefore, such a particle sliding from rest at a point P be not stopped by some extraneous force when it comes to K, the lowest point in the curve, it must evidently proceed to ascend the other side of the curve KB, until the velocity which it had at K be entirely destroyed by the action of gravity; which will be the case when it has risen to a point P', whose vertical height above K is the same as that of P.

For an instant, then, the particle will be at rest, but will immediately afterwards commence sliding down $P'K$, and from

thence up to KP again. If nothing be altered in the circumstances of the particle, it will repeat these alternate movements *ad infinitum*, and is usually said to *oscillate* from P to P'.

If the curve be symmetrical on both sides of the vertical through K, it is not difficult to see that each length, PK, $K'P$, must be described in the same time, whether the particle be moving up it or down it; and therefore the *time of oscillation* between P and P' must be the same each way. Generally this interval will vary for different positions of P in the same curve; but there is one particular curve, called the *cycloid*, in which it is the same wherever in the curve P may be chosen: this is what is meant when it is said that the oscillations of a particle in a cycloid are *isochronous*.

59. If a circle, as AMB, roll along a straight line ZAL, any fixed point in it, as M, will trace out the curve which is denominated the *cycloid*. If ZL be horizontal, and the circle roll along below it, the curve will be generated in the position required for the isochronous oscillation of a particle along it.



Let AB be the vertical diameter of this generating circle in any given position, K the vertex or lowest point of the cycloid, and KL vertical; the curve will be symmetrical upon both sides of KL, and, by a known property of the curve, the arc KM will equal twice the chord BM of the circle.

Now, if MM' be drawn perpendicular to KL, meeting AB in m, we have

$$BM^2 = BA.Bm = KL.KM';$$

therefore $$KM = 2\sqrt{(KL.KM')}.$$

Suppose a heavy particle to be sliding down this cycloid under the action of gravity, and let P be the point from

which it started from rest: draw PP' perpendicular to KL; then the velocity of the particle upon arriving at M will be

$$=\sqrt{(2gP'M')}.$$

If N be a point in the curve indefinitely near M, we may suppose the particle to move from M to N uniformly with the velocity which it had at M, and therefore its time of passing from M to N

$$=\frac{MN}{\sqrt{(2gP'M')}}$$

$$=\frac{KM-KN}{\sqrt{(2gP'M')}}$$

$$=\frac{2\sqrt{(KL.KM')}-2\sqrt{(KL.KN')}}{\sqrt{(2gP'M')}}$$

$$=\sqrt{\left(\frac{2KL}{g}\right)}\left\{\sqrt{\left(\frac{KM'}{P'M'}\right)}-\sqrt{\left(\frac{KN'}{P'M'}\right)}\right\}.$$

Upon $P'K$ as diameter describe a semicircle, cutting MM' and NN' in n and n' respectively; join $P'n$, $P'n'$, and also Kn, Kn', and let $P'n'$ meet Kn in o.

Then $$\sqrt{\left(\frac{KM'}{P'M'}\right)}=\sqrt{\left(\frac{KM'.KP'}{P'M'.KP'}\right)}$$

$$=\frac{Kn}{P'n},$$

and similarly $$\sqrt{\left(\frac{KN'}{P'M'}\right)}=\frac{Kn'}{P'n}.$$

Therefore time of moving from M to N

$$=\sqrt{\left(\frac{2KL}{g}\right)}\frac{Kn-Kn'}{P'n}.$$

Since n and n' are indefinitely near to each other, the angle $nP'o$, and therefore $n'Ko$, is indefinitely small; and therefore, since the angles at n and n' are both right angles, the angles at o, i.e. $P'on$ and Kon', must be approximately right angles also; therefore $Kn'=Ko$ approximately, and $Kn-Kn'=on$.

Hence time from M to $N = \sqrt{\left(\frac{2KL}{g}\right)} \frac{on}{P'n}$

$$= \sqrt{\left(\frac{2KL}{g}\right)} \times (\text{angle } nP'n').$$

The same may be proved of the time of falling between any two points of the curve indefinitely close to each other; therefore, adding all these times together,

$$\text{time of falling from } P \text{ to } K = \sqrt{\left(\frac{2KL}{g}\right)} \times \phi,$$

where ϕ represents the sum of all the angles which the arcs, into which the semicircle is divided, corresponding to the successive small arcs of the cycloid, subtend at P': but the sum of these angles is the same as the angle which the arc of the whole semicircle subtends at $P' = \frac{\pi}{2}$.

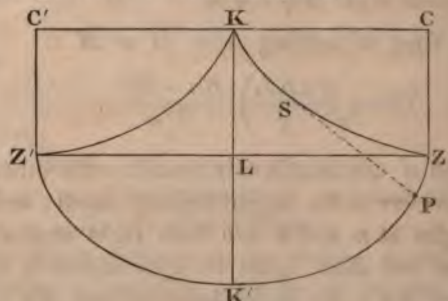
Hence time of passing from P to $K = \sqrt{\left(\frac{2KL}{g}\right)} \cdot \frac{\pi}{2}$;

and, because the time of ascending the opposite side must be the same as this, for the reasons before hinted at,

$$\text{time of entire oscillation} = \pi \sqrt{\left(\frac{2KL}{g}\right)},$$

which is evidently independent of the position of P, and therefore proves the proposition.

60. We will now proceed to shew how a particle may be made practically to oscillate in a cycloid.



Taking the same figure as in Art. (58), let the axis $K'L$ be produced to K, so that $LK = LK'$; complete the parallelogram $KLZC$, and upon ZC as axis describe the semicycloid KSZ: this will be similar and equal in every respect to the portion ZPK' of the original cycloid. In the same way let a semicycloid KZ' be formed, symmetrical with KZ, on the other side of KK'.

It is a known property of the cycloid, that if a string having one extremity fastened at K, and being wrapped upon KZ, be unwrapped, beginning from the end Z in such a way as always to be kept stretched, its extremity, as P, which was at Z, will trace out the cycloid ZK': for the same reason, of course if P be carried beyond K', as the string wraps itself upon KZ', P will describe the curve $K'Z'$.

It is very evident, that if a heavy particle be attached to the string at P, gravity will make it keep the string stretched; and therefore if the particle be allowed to fall from any initial position, as for instance P in the figure, it will necessarily oscillate in a portion of the cycloid $ZK'Z'$, describing equal arcs on both sides of K'.

What we have shewn with regard to the oscillation of a particle upon a *smooth* curve, went upon the supposition that gravity was the only force acting upon the particle, which was not normal to the path described; now, in the case before us, where the particle attached to a string is made to describe a cycloid, the tension of the string is always in the direction of the normal to the curve, and gravity is the only other force acting: hence we have, as before,

$$\text{time of oscillation} = \pi \sqrt{\left(\frac{2K'L}{g}\right)}$$

$$= \pi \sqrt{\left(\frac{KK'}{g}\right)};$$

or putting T to designate the time of oscillation, and calling the length of the string l, which is clearly the same as KK',

$$T = \pi \sqrt{\frac{l}{g}} \quad \ldots\ldots\ldots\ldots\ldots \text{(A)}.$$

A particle oscillating in this manner is usually termed a *pendulum*.

It may not be without use to remark here, that the lengths l and g must always be measured in the same unit of length; and that T must be estimated in terms of the same unit of time as that which determines the value of g.

61. In making experiments with a pendulum of known length, the duration of an oscillation, and therefore the value of T, can be very accurately observed; the value of g can then be deduced by equation (A) of Art. (60). This method of determining g is far more correct than that afforded by Atwood's machine, on account of the difficulty in the latter of determining the effect of the pulleys in retarding the motion.

By this means it is found that the value of g varies at different parts of the earth's surface, according to a law which depends upon the latitude of the place, but that it is always the same at the same place.

In the latitude of Greenwich the length of the pendulum whose duration of oscillation is one second, or of the *seconds' pendulum*, as it is commonly called, is 39.1393 inches; it becomes shorter in latitudes nearer the equator, which apparently shews that the force of gravity diminishes.

The pendulums which are used to regulate the going of clocks are commonly seconds' pendulums.

Examples to Section VI.

(1). A pendulum which vibrates seconds at the earth's surface, is found to lose s seconds in t hours when taken to the bottom of a mine; find the depth of the mine, supposing the accelerating force of gravity to vary as the distance from the centre of the earth.

If l be the length of the pendulum, g the force of gravity at the surface of the earth, g' at the bottom of the well;

$$\text{duration of an oscillation at the surface} = \pi \sqrt{\frac{l}{g}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ in the mine} = \pi \sqrt{\frac{l}{g'}};$$

therefore
$$\frac{t}{60 \times 60 \times t} : \frac{t}{60 \times 60 \times t - s} :: \frac{1}{\sqrt{g}} : \frac{1}{\sqrt{g'}},$$

or
$$\sqrt{\frac{g}{g'}} = \frac{3600t}{3600t - s}.$$

If now r be the radius of the earth, z the depth of the mine, we have by our supposition

$$\frac{g}{g'} = \frac{r}{r - z}.$$

Hence, by substitution,

$$1 - \frac{z}{r} = \left(1 - \frac{s}{3600t}\right)^2,$$

therefore
$$z = \frac{2s}{3600t} r \text{ nearly.}$$

General Examples.

(1). If 10 minutes were taken as the unit of time, and the numerical value of the accelerating force of the earth's attraction were 40, what would be the unit of length?

(2). One railway train travels uniformly 20 miles in 40 minutes, and a second 40 miles in an hour-and-a-half; compare the velocities of the trains.

(3). A man sets out to swim directly across a river whose breadth is b; when he arrives at the other side, he is at a distance a below the point from which he started: shew that he will regain this point if he incline his course back again at an angle θ to the stream, such that

$$\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{a}{b}.$$

Interpret the result when a is = or > b.

(4). A particle acted upon by a constant force has a velocity with which, if it were free, it would describe 300 feet in a minute; and after 3 seconds it has a velocity of 2 feet per second: find the force referred to a minute as the unit of time.

(5). Generally the velocity due to the force is $\sqrt{(2fs)}$, which is independent of the initial velocity u, therefore the whole velocity $= u + \sqrt{(2fs)}$: point out the fallacy of this reasoning.

(6). A ball moving at the rate of 1000 feet per second loses $\frac{2}{3}$ of its velocity in passing through a three-inch board; how long is it on its passage (supposing the resistance uniform)?

(7). Prove that the velocity acquired by a body in falling from rest from A to B, is a mean proportional between the sum and difference of the velocities with which a body in falling from any height passes the points A and B.

(8). A body falling to the ground is observed to pass through $\frac{8}{9}$ of its original height in the last second; find the height.

(9). A body is projected vertically upwards with a velocity $2g$; find the velocity with which another body must be projected vertically upwards one second afterwards, so as to strike the first at its highest point of elevation.

(10). Two bodies are allowed to slide down an inclined plane from the same point, with an interval of $1''$ between the times of starting; compare their distances from one another after $1''$, $2''$, &c.

(11). A heavy body has fallen from A to B when another body is let fall from a lower point C in the same vertical line; how far will the latter body fall before it is overtaken by the former?

(12). A constant force acts upon a body during $3''$ from rest and then ceases. In the next $3''$ it is found that the body describes 180 feet. Find the velocity of the body at the end of the second second of its motion. Also find the accelerating force, $1°$ when a second, $2°$ when a minute is taken as the unit of time.

(13). Two circles lie in the same plane, the lowest point of one being in contact with the highest point of the other; shew that the time of descent from any point of the former to a point in the latter, down the chord passing through the point of contact, is constant.

(14). In a vertical parabola a tangent is drawn at any point P cutting the axis produced in T; shew that if gravity alone acts, the time of descent down TP bears a constant ratio to the time of descent from T to the focus.

(15). Given the time of flight of a projectile on a horizontal plane and its initial velocity, find the greatest height to which it rises.

(16). An arrow shot upwards attains a height of 200 feet: find the greatest horizontal distance the arrow may be shot with the same force.

(17). The length of base of a pyramid is 130 feet, its height 68; find the direction and least velocity with which a person standing at its foot may throw a stone over the top.

(18). A body shot upwards is carried with uniform horizontal motion by the wind; determine its path and the point at which it falls.

(19). A body is projected horizontally with a velocity 39; find the position of the focus of the parabola described, and the direction of the body's motion after $3''$.

(20). The time of describing any portion PQ of the parabolic path of a body acted upon by gravity, is proportional to the difference of the tangents of the angles which the tangents at P and Q make with the horizon.

(21). When a body is projected from a given point of an inclined plane, find the greatest distance from the inclined plane which it attains, and its velocity in that position.

(22). A balloon is floating at a known altitude, with a horizontal velocity V; a ball is let fall from it, and is observed to strike a vertical wall at a height 6 from the ground: determine the angle at which it meets the wall.

(23). If a stone be dropped from the top of a tower at the equator, determine the distance from the foot of the tower of the point where it falls.

(24). Find the rate at which a carriage is travelling when the dirt thrown from the rim of the wheel to the greatest height reaches a given level.

(25) A body slides down an inclined plane and raises another by a string over the highest point of the plane; find the tension of the string. If the string at any instant were to break, what would be the subsequent motion of the bodies?

(26). A body descending vertically draws an equal body 36 feet in $3''$ up a plane inclined $30°$ to the horizon; how far did it fall in the 1st second, and how far would it fall in the nth?

(27). A constant force p, acting upon a body, generates a velocity u in t seconds; find the mass of the body.

(28). Two weights, each of 7 lbs., hang by means of a string, whose weight may be neglected, over a fixed pulley; to one of them an additional weight of 2 lbs. is added, which is removed after the weight has fallen through 8 feet: find the time occupied in falling through the 8 feet, and also the time in which 8 feet would be described after the removal of the additional weight.

(29). A railway-train is going smoothly along a curve of 500 yards' radius at the rate of 30 miles an hour; find at what angle a plumb-line hanging in one of the carriages is inclined to the vertical.

(30). A carriage of given weight moves on a railroad with a known velocity round a curve of known radius; find the amount by which the outer rail must be elevated above the inner one, in order that the carriage should not be overturned. (Position of the centre of gravity of the carriage given.)

(31). A string will just bear a weight of 16 lbs. without breaking: if a weight of $\frac{1}{2}$ a lb. be attached to it, and whirled round in a horizontal circle whose radius is 2 feet, find the number of revolutions the weight must make in a second in order to break the string.

(32). How does centrifugal force affect the apparent weight of a body at different parts of the earth's surface?

(33). From what height must a weight of 12 lbs. fall, in order that it may impinge upon the ground with the same effect as a weight of 25 lbs. falling 6 feet?

(34). A number n of equal inelastic balls are placed in a row on a horizontal plane. The first impinges on the second with a velocity of a feet per second, and the second impinges upon the third and so on. Find the velocity communicated to the last.

What will be the time which elapses before the last body is put in motion, the balls being at equal distances b from each other?

(35). A, B, C are the masses of three perfectly elastic balls in the order of their magnitudes. A strikes B at rest with a given velocity and drives it against C, the distance between B and C being given, and also the velocity of A; find when A will overtake B.

(36). Two imperfectly elastic bodies A and B, moving with equal velocities in opposite directions, impinge directly. Shew that if after impact A is at rest and B moves back with the velocity it had before impact, then the mass of A is twice that of B.

(37). When two bodies impinge upon one another, shew that the velocity of their centre of gravity is the same before and after impact.

(38). A given elastic body is thrown upwards; at what point must a hard horizontal plane be interposed, in order that the body may reach the ground again with the velocity with which it impinges upon the plane?

(39). A ball, whose elasticity is $\frac{1}{4}$, falls on a horizontal plane from a height of 16 feet; how long will it be before it has ceased to move (assume gravity $= 32$)?

(40). Two vertical planes are separated by an interval a. If a perfectly elastic ball be projected from one of them in a horizontal direction towards the other, with a velocity due to the height a, it will, after reflexion, strike the first plane at a distance a below the point of projection.

(41). A body is projected with a given velocity along a smooth horizontal plane, it meets a small inclined plane whose intersection with the horizontal plane is perpendicular to its motion; find the inclination of the plane, 1st, when it causes the body to bound to the greatest distance along the horizontal plane, 2nd, to the greatest height.

(42). A perfectly elastic ball is projected upon a plane. If the angle of projection be 60°, and the inclination of the plane 30°, shew that, after rebounding from the plane, the ball will ascend vertically.

(43). A perfectly elastic ball fell upon an inclined plane at A, rebounded, and again struck the plane at B. If AB equal a, and the inclination of the plane be α, shew that the height from which the body fell

$$= \tfrac{1}{8} a \operatorname{cosec} \alpha.$$

(44). A perfectly elastic ball is projected towards a distant wall, in a direction making 45°, with a smooth horizontal plane, and with a velocity which would be acquired in falling down $2\frac{2}{3}$ the height of the wall; compare the chances of the ball bounding over or striking the wall.

(45). Find the length of a simple pendulum which vibrates 90 times in a minute.

(46). A clock loses 5 minutes per diem: how much must its pendulum be shortened in order that the error may be corrected? given that the length of a seconds' pendulum is equal to 39.14 inches.

(47). A seconds' pendulum carried up to the top of a mountain, is found to lose $43''.2$ a day; find the height of the mountain, supposing gravity to vary inversely as the square of the distance from the centre of the earth, and the radius of the earth to be 4000 miles.

(48). The number of vibrations of two pendulums, whose lengths are l and l' in the same time at different places are as $m : m + m'$; determine the proportion of the force of gravity at these two places.

(49). Two pendulums, whose lengths are l and l', begin to oscillate at the same time and are again coincident after n oscillations of the former pendulum. Having given l, it is required to determine l'.

(50). A railway-train is moving smoothly along a curve at the rate of 60 miles an hour, and in one of the carriages a pendulum, which would ordinarily oscillate seconds, is observed to oscillate 121 times in 2 minutes; shew that the radius of the curve is very nearly 2 furlongs.

Suppose a stone to be dropped from the window of this carriage, how far from the rail would it fall?

THE END.

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